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On First-Order Corrections to the LSW Theory I: Infinite Systems

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We present a new method to efficiently identify the first-order correction to the classical model by Lifshitz, Slyozov and Wagner (LSW). The latter describes the evolution of second phase particles embedded in a matrix during the last stage of a phase transformation and is valid in the regime of vanishing volume fraction ϕ of particles. We consider a statistically homogeneous (and thus infinite) system, where the first-order correction is of order $\phi^{1/2}$. The key idea is to relate the full system of particles to systems where a finite number of particles has been removed. This method decouples screening and correlation effects and allows to efficiently evaluate conditional expected values of the particle growth rates.

KEY WORDS: Ostwald ripening; coarsening rates; correlations.

1. SUMMARY AND INTRODUCTION

1.1. Summary

The classical theory by Lifshitz and Slyozov[(14)] and Wagner[(35)] (LSW) describes the last stage of a phase transition, where precipitate particles in a melt undergo competitive growth, known as Ostwald ripening. We refer to Section 1.2 for a detailed scenario. The classical LSW theory predicts how the distribution of radii evolves; in particular, it predicts the growth rate of the average particle size. The LSW theory is introduced in

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Section 1.5. It is based on the postulate that the particles only communicate via a single mean-field. This is a good approximation of reality only in the regime where the effective range of interaction between the particles, given by the screening length, is much larger than their typical distance. The latter is true in the regime of vanishing volume fraction ϕ of the particles. The screening length and the validity of the LSW theory as a zero-order theory is discussed in Section 2.1.

It is extremely convenient that the evolution of the complex system with finite interactions closes on the level of one-particle statistics in the limit of vanishing ϕ. But the quantitative predictions of the LSW theory deviate from standard experiments (see e.g. ref. 10). It is generally conjectured that this deviation is due to the fact that ϕ is small but finite. Hence in order to extend the range of validity of the convenient LSW theory, it is of major interest to identify a first-order correction in ϕ, which still closes on the level of – at most – two-particle statistics. This requires an asymptotic analysis of the complex model with finite interactions in a statistical framework. The usual starting point is the monopole approximation of the evaporation–recondensation mechanism. The evaporation–recondensation mechanism is introduced in Section 1.3 and the monopole approximation is formally derived in Section 1.4.

The main goal of this paper is to present a novel approach to analyze statistically homogeneous and thus infinite system. In this case, the first-order correction to the LSW theory is of order $\phi^{1/2}$. This is in contrast to finite systems with size smaller than the screening length. Here the first-order correction is of order $\phi^{1/3}$. We formally derive the screening length in Section 2.1 and give a heuristic argument for the $\phi^{1/3}$ scaling, when the system is smaller than the screening length (Section 2.2). Then we turn to infinite systems and report on the work of Marqusee and Ross[17], of Tokuyama *et al.*[29], and of Marder[16] in Section 2.3.

In Section 3 we present the main results of this paper, which are proved in the remaining Sections 4–8. We propose a new method for identifying conditional expectations of particle growth rates up to the order $\phi^{1/2}$. The key idea is to relate the full particle system to the system with a finite number of particles removed. The strategy is similar in spirit to the so-called method of reflection or Schwarz alternating method, which has also been used in ref. 12 for example. With the help of this method we recover the first-order correction of Marder for the two-particle statistics (Section 3.2.2). As a byproduct we obtain also the first-order correction of Marqusee and Ross for the one-particle statistics (Section 3.2.1). The conceptual advantage of this method is that it decouples screening and correlation effects in the first-order correction. Moreover, we will derive Marder's result under more natural assumptions on the statistics of the particles (cf. the discussion in Section 3.2.2). We also show the self-consistency of our statistical assumptions in Section 3.2.3.

In a companion paper[11], to which we refer in the following as Part II, we consider systems with a finite number of particles, where the above argument for a $\phi^{1/2}$ scaling does not apply. Indeed, Fradkov *et al.*[9] and Mandyam *et al.*[15] have numerically observed a crossover in the scaling of the first-order correction term for finite systems. As indicated before, it changes from $\phi^{1/3}$ to $\phi^{1/2}$ when the system becomes larger than the screening length. By varying the number n of particles at given volume fraction ϕ we theoretically establish this crossover in Part II by a variational argument using the monopole approximation.

1.2. Coarsening

The last stage of a first-order phase transformation is characterized by a coarsening of the morphology of the phase distribution. We think for instance of the Cahn–Hilliard model, which describes the spinodal decomposition of a homogeneous two-component mixture which is quenched into the unstable region. Then there are two equilibrium values for the concentration of the A-component (and the B-component). If the total volume fraction of the A-component is sufficiently small, many "nuclei" or "particles" of the phase with the higher equilibrium concentration form. They are immersed in a background phase of the lower equilibrium concentration, the "matrix". Hence the system is close to equilibrium in the bulk, but the interfacial layer carries a specific surface energy. Because of the conservation of the total volume of A- and B-components, the volume fraction ϕ occupied by the particles (as opposed to the matrix) is preserved. Hence the total surface energy can only be reduced if the large particles grow at the expense of the smaller ones, which eventually vanish. This competitive growth process, which is limited by the diffusion of A-atoms through the matrix, is known as Ostwald ripening.

1.3. The Evaporation–Recondensation Mechanism

In this section, we recall the simplest model which captures the late stage coarsening through evaporation and recondensation of A-atoms: the Mullins–Sekerka model, see ref. 26 for a derivation from the Cahn–Hilliard model. In this model, the interfacial layer is replaced by a sharp interface. It is further based on the assumption that the movement of the interface is so slow that the diffusion field u is in quasistatic equilibrium. Equilibrium in the bulk is expressed in Eq. (1.2) below, whereas equilibrium on the interface leads to the Gibbs–Thomson condition (1.3) in suitably non-dimensionalized variables. These two conditions are supplemented by the kinematic Stefan condition (1.1):

$$V = [\nabla u \cdot \vec{n}] \quad \text{on the interface,} \tag{1.1}$$

$$-\Delta u = 0 \quad \text{in the bulk,} \tag{1.2}$$

$$u = \kappa \quad \text{on the interface.} \tag{1.3}$$

Here κ denotes the mean curvature[4] of the interface, which can be understood as measuring the "exposure" of the A-atoms at the interface; $\vec{n}$ is the normal to the interface pointing into the majority phase, $[\nabla u \cdot \vec{n}]$ the jump of the normal component of the gradient across the interface and V the normal velocity of the interface. As desired, this model preserves the total volume covered by each phase while it reduces total interfacial area.

1.4. The Monopole Approximation

In this section, we recall the monopole approximation of the Mullins–Sekerka model (1.1)–(1.3), as introduced by Weins and Cahn[36]. We are interested in the regime where the volume fraction ϕ of the particles is very small, which in particular implies

$$\text{typical particle distance} \gg \text{typical particle radius.} \tag{1.4}$$

Then the particles P_i are approximately balls with radius R_i and immobile center X_i (this is worked out in a rigorous manner in refs. 2 and 3, cf. also ref. 34). In view of (1.1) and (1.2), a natural Ansatz for u is

$$u(x) = u_\infty + \sum_j \frac{B_j}{|x - X_j|}, \qquad x \in \mathbb{R}^3 \setminus \cup_i P_i, \tag{1.5}$$

where we denote by $\{4\pi B_i\}_i$ the negative growth rates of the particle volumes, that is

$$-B_i := \frac{d}{dt}\left[\frac{1}{3} R_i^3\right] = R_i^2 \frac{dR_i}{dt}. \tag{1.6}$$

[4]With the convention that the curvature of a ball is positive.

for radially symmetric particles. The constant u_∞ is called the "mean field". The volume conservation translates into

$$\sum_i B_i = 0. \tag{1.7}$$

The physics literature usually appeals to electrostatic intuition: One thinks of the particles as conductors. Then B_i and $\frac{1}{R_i}$ correspond to the total charge resp. the potential of the ith particle.

A priori, (1.5) is a good approximation away from the particles. Since the particles are expected to be nearly radially symmetric, (1.5) is also a good approximation close to the, say, ith particle:

$$u(x) = u_\infty + \frac{B_i}{R_i} + \sum_{j\neq i} \frac{B_j}{|x - X_j|} \overset{(1.4)}{\approx} u_\infty + \frac{B_i}{R_i} + \sum_{j\neq i} \frac{B_j}{d_{ij}} \quad \text{on} \quad \partial P_i,$$

where $d_{ij} := |X_i - X_j|$ is the distance between particle centers. Using the Gibbs–Thomson condition (1.3) we may view the mean field u_∞ and the growth rates $\{B_i\}_i$ as the solution of the linear system of equations

$$\frac{1}{R_i} = u_\infty + \frac{B_i}{R_i} + \sum_{j\neq i} \frac{B_j}{d_{ij}} \tag{1.8}$$

under the constraint (1.7). Observe that u_∞ can be interpreted as a Lagrange multiplier for (1.7). It has been argued that the error coming from the monopole approximation is of higher order in ϕ than the first-order correction to the LSW theory. Indeed, the error is of order $\phi^{2/3}$ as can be deduced e.g. from Eq. (2.42) of ref. 1 Consequently, we allow ourselves to neglect in the upcoming analysis contributions which come from dipolar terms.

1.5. The Classical LSW Theory

In this section, we recall the classical LSW theory, as introduced by Lifshitz and Slyozov[(14)] and Wagner[(35)]. The solution of the classical LSW theory, which we denote by $\{B_i^{\mathrm{LSW}}\}_i$, is given by the truncation of (1.8)

$$\frac{1}{R_i} = u_\infty^{\mathrm{LSW}} + \frac{B_i^{\mathrm{LSW}}}{R_i}, \tag{1.9}$$

which together with (1.7) yields

$$B_i^{\mathrm{LSW}} = 1 - R_i\, u_\infty^{\mathrm{LSW}} \quad \text{and} \quad u_\infty^{\mathrm{LSW}} = \frac{\sum_i 1}{\sum_i R_i}. \tag{1.10}$$

In particular, the LSW mean field is given by the inverse of the mean radius of particles. It is then natural to pass to a continuum description. Let $f(R)$ denote the empirical distribution of particle radii:

$$\int_{R_-}^{R_+} f(R)\, dR = \#\{ i \mid R_i \in (R_-, R_+) \}.$$

Without further approximations, (1.9) and (1.6) now turn into

$$\frac{\partial f}{\partial t}(t, R) - \frac{\partial}{\partial R}\left(\frac{1}{R^2}\,(1 - R\, u_\infty(t))\, f(t, R) \right) = 0, \tag{1.11}$$

and (1.10) can be written as

$$u_\infty = \frac{\int_0^\infty f\, dR}{\int_0^\infty R\, f\, dR}. \tag{1.12}$$

Hence we have an evolution law for the empirical distribution of radii.

A scale invariance of (1.11) and (1.12) suggests that the number of particles decreases as t^{-1}, whereas their mean radius increases as $t^{1/3}$. In fact, the evolution (1.11) and (1.12) allows for a one-parameter family of self-similar solutions. Based on formal arguments, LSW predict that, independently of the initial data, all solutions converge toward a particular one of the above self-similar profiles (see ref. 22 for a rigorous mathematical analysis, which shows that in general universal asymptotics cannot be expected). As a consequence, LSW obtain for the mean radius

$$\frac{1}{u_\infty} \approx \left(\frac{4}{9}\, t \right)^{1/3}. \tag{1.13}$$

Experiments on Ostwald ripening show the same exponent but considerably larger growth rates than given by all self-similar solutions. The general belief is that the deviation is due to the finiteness of ϕ, as has been pointed out already in ref. 4. The LSW theory treats the spatial arrangement as if particles are infinitely far away, which overestimates the distance over which particles have to diffuse and thus the constant in (1.13) underestimates the coarsening rate.

2. SCALING ARGUMENTS

2.1. The Screening Length

We now formally uncover the screening effect starting from (1.8). Let us introduce $\{u_i\}_i$ via

$$\frac{1}{R_i} = u_i + \frac{B_i}{R_i}, \quad \text{that is} \quad B_i = 1 - R_i\, u_i. \tag{2.1}$$

As opposed to the LSW truncation, $\{u_i\}_i$ may not be a constant. We replace $\{B_i\}_i$ in (1.8) according to (2.1) and obtain

$$0 = u_\infty - u_i + \sum_{j\neq i} \frac{1}{d_{ij}}\,(1 - R_j\, u_j). \tag{2.2}$$

Let us now, as in Section 1.5, pass to a continuum description, which as opposed to before has also a spatial resolution. Let $f(x, R)$ denote the number density of particles of given center position and radius, i.e. given a bounded volume Ω and an interval of radii (R_-, R_+) we have:

$$\int_{R_-}^{R_+} \int_\Omega f(x, R)\, d^3x\, dR = \#\{\, i \mid (X_i, R_i) \in \Omega \times (R_-, R_+)\,\}.$$

Since (2.2) can be written as

$$u_i = \sum_{j\neq i} \frac{1}{d_{ij}}\,(1 - R_j\, u_j) + u_\infty,$$

we expect $\{u_i\}_i$ to be only slowly varying in space, which we indicate by writing $u_i = u(X_i)$. Hence (1.6) and (2.1) turn into

$$\frac{\partial f}{\partial t}(t, x, R) - \frac{\partial}{\partial R}\left(\frac{1}{R^2}\,(1 - R\, u(t, x))\, f(t, x, R)\right) = 0, \tag{2.3}$$

which has the same form as (1.11) but contains x as a parameter.

We now turn to (2.2). Its continuum version is, taking into account that u is slowly varying,

$$\begin{aligned} 0 &= u_\infty - u(x) + \int\int \frac{1}{|x-y|}\,(1 - R\, u(y))\, f(y, R)\, d^3y\, dR \\ &= u_\infty - u(x) + \int \frac{1}{4\pi\, |x-y|}\,(4\pi\, \rho(y) - \mu(y)\, u(y))\, d^3y, \end{aligned} \tag{2.4}$$

where ρ and μ are the number resp. capacity density of the particles, that is

$$\rho(x) := \int_0^\infty f(x, R)\, dR \quad \text{and} \quad \mu(x) := \int_0^\infty 4\pi\, R\, f(x, R)\, dR. \quad (2.5)$$

We call μ the capacity density since $4\pi R$ is the capacity of a ball of radius R. We now apply the Laplace operator to the identity (2.4) and obtain

$$-\triangle u(x) + \mu(x)\, u(x) = 4\pi\, \rho(x). \quad (2.6)$$

In the language of electrostatics, this equation for the mesoscopic potential $u(x)$ displays the effective screening in our arrangement of charged particles. In contrast to (1.12), Eq. (2.6) highlights that particles interact only over a finite length, called the screening length, which is related to the average capacity density $\bar{\mu}$ via

$$\begin{aligned} \text{screening length} : = \frac{1}{\sqrt{\bar{\mu}}} &\sim \frac{(\text{typical particle distance})^{3/2}}{(\text{typical radius})^{1/2}} \\ &\gg \text{typical particle distance}, \end{aligned} \quad (2.7)$$

where the last inequality follows from (1.4). As can be seen from (2.7), this effective interaction range includes indeed many particles in the regime $\phi \ll 1$. The screening length also sets the relevant length scale in (2.6) and therefore determines the length scale over which u varies. This *a posteriori* legitimates our assumption that $\{u_i\}_i$ is slowly varying in space.

Let us now discuss when (1.9) can be expected to be a *zero-order* approximation of (1.8). Analogously to the LSW theory, (2.3), (2.5) and (2.6) define a closed time evolution of the number density $f(x, R)$. We observe that this evolution projects onto (1.11) and (1.12) for $f(R) = \int f(x, R)\, d^3x$ if $u(x)$ is spatially constant. Hence (1.9) is a zero-order approximation of (1.8) provided $u(x)$ is approximately spatially constant. This is the case if one of the two following scenarios are true.

(S1) If

$$\text{system size} \ll \text{screening length}. \quad (2.8)$$

(S2) If the empirical distributions $\rho(x)$ and $\mu(x)$ are (statistically) homogeneous on length scales of the screening length. Because of (2.7), this is the case if

$$\text{system size} \gg \text{screening length}$$

and if $\rho(x)$ and $\mu(x)$ are statistically homogeneous in the interior of the system.

Therefore, it is no surprise that there are two different first-order corrections to LSW.

The first rigorous derivation of (1.11) and (1.12) starting from (1.1), (1.2) and (1.3) can be found in refs. 19 and 20 in the regime (2.8). An analysis in ref. 21, 23 and 24 derives (2.3), (2.5) and (2.6) from (1.1), (1.2) and (1.3) in the general case, and thus makes the above argument in favor of a screening length rigorous.

2.2. Heuristic Argument for the $\phi^{1/3}$ Scaling

In this section, we give an argument for the $\phi^{1/3}$ scaling of the first-order correction term in case of Scenario (S1) from Section 2.1.

The $\phi^{1/3}$ scaling is much easier to uncover than the $\phi^{1/2}$ scaling. One just treats $\sum_{j\neq i}\frac{B_j}{d_{ij}}$ as a small perturbation in (1.8). Rewriting (1.8) as

$$1-R_i\,u_\infty = B_i-\sum_j g_{ij}\,B_j,$$

where the matrix $\mathbf{g}=\{g_{ij}\}_{ij}$ is given by

$$g_{ij}=-\frac{R_i}{d_{ij}}\quad\text{for } j\neq i\quad\text{and}\quad g_{ii}=0, \tag{2.9}$$

we observe that this is justified if the matrix $\mathbf{g}$ is "small enough". Taking for instance the matrix norm corresponding to the maximum norm,

$$\begin{aligned}\|\mathbf{g}\|=\sup_i\sum_j|g_{ij}| &\sim \frac{\text{typical radius}\times(\text{system size})^2}{(\text{typical distance})^3}\\ &\sim\left(\frac{\text{system size}}{\text{screening length}}\right)^2,\end{aligned}$$

where we used (2.7) in the latter equality, we observe that this is definitely true in the regime (2.8), which corresponds to the Scenario (S1) from Section 2.1. This point of view suggests an asymptotic development by a Neumann series:

$$B_i=(1-R_i\,u_\infty)+\sum_j g_{ij}\,(1-R_j\,u_\infty)+\sum_j\sum_k g_{ij}\,g_{jk}\,(1-R_k\,u_\infty)+\cdots \tag{2.10}$$

where u_∞ has to be determined such that (1.7) holds to the desired order. Since the entries of the matrix are proportional to $\phi^{1/3}$ and the matrix is multiplied with vectors having zero average, this yields an expansion in $\phi^{1/3}$ (a more detailed investigation of the subcritical case will be given in ref. 6).

2.3. $\phi^{1/2}$ Scaling, State of the Art

In this section, we review the various derivations of a first-order correction in $\phi^{1/2}$ in the physics literature.

Marqusee and Ross[(17)] assume that at any time t, $\{(R_i, X_i)\}_i$ are statistically independent and identically distributed (a property which would be preserved by the LSW dynamics). Starting from the monopole approximation (1.8), they identify the evolution of the one-point statistics. They do this by manipulating the non-convergent series (2.10), which they interpret as a multiple scattering series. They so obtain a correction to (1.11) and (1.12) of order $\phi^{1/2}$, which is a consequence of screening effects. In the second part of their paper, Marqusee and Ross analyze the perturbation of the self-similar solution of (1.11) and (1.12) by their first-order correction and find that the expected radius grows as

$$\langle R\rangle = \left(\frac{4}{9}t\right)^{1/3}\left[1+0.740\,\phi^{1/2}+O(\phi)\right],$$

which is to be compared with (1.13).

It is obvious that the assumption that $\{(R_i, X_i)\}_i$ are statistically independent is not preserved by the evolution: A medium sized particle in the neighborhood of a large particle will shrink faster than in an average environment. Hence a large particle eventually influences the statistics of $\{(R_i, X_i)\}_i$ within the screening length. This in turn will influence the evolution of that large particle.

Marder[(16)] realized that this effect leads to an additional correction term of the same order $O(\phi^{1/2})$. He shows this by deriving the evolution of the two-point statistics up to an error $o(\phi^{1/2})$. His approach is motivated by statistical mechanics and does not rely on (2.10). Starting from the monopole approximation (1.8) (with a physically motivated but mathematically immaterial truncation) he generates a hierarchy of equations for the expectation value of B_1 *conditioned* on the position and radius $(R_1, X_1), \ldots, (R_k, X_k)$ of a finite number of particles. He truncates the hierarchy on the level of two-particle statistics by a closure hypothesis (ref. 16, Section II.C).

In the second part of the paper, Marder performs an analysis of the evolution for the two-point statistics. He assumes that particles are initially independently distributed and then linearizes around the Marqusee–Ross theory. The resulting equations are solved numerically. As an effect of correlations, Marder's theory predicts a significantly stronger broadening of the self-similar particle size distribution than the Marqusee–Ross theory.

Yet a different calculus has been developed in Tokuyama, Enomoto and Kawasaki[13,28–31]. They also start from the monopole approximation (1.8) but allow for arbitrary correlations. They avoid using the Neumann series in the form of (2.10) with help of a method developed by Mori *et al.*[18]: They split the matrix $\mathbf{g}$ (cf. (2.9)) into its expectation value $P\,\mathbf{g}$ and the fluctuating part $Q\,\mathbf{g}$. This leads to a new Neumann series in $Q\,\mathbf{g}^T\,(\mathbf{1}-P\,\mathbf{g}^T)^{-1}$, where the superscript T denotes the transpose. To study its convergence properties, a diagrammatic representation is used. Like Marqusee and Ross, they obtain a first-order correction in $\phi^{1/2}$ to (1.11) and (1.12), which in addition contains a not very explicit term coming from correlations.

3. MARDER AND MARQUSEE-ROSS THEORY REVISITED

In the remainder of this paper we will rederive Marder's evolution for the one- and two-particle statistics of $\{(R_i, X_i)\}_{i\geqslant 1}$. We start from a more natural closure assumption than his. As a byproduct, we obtain a simple derivation of Marqusee–Ross's evolution for the one-particle statistics.

The main step is to rederive the expression for $\langle B_1|(R_1, X_1), (R_2, X_2)\rangle$, the expected value of the growth rate of particle 1 conditioned on particles 1 and 2. We assume that the distribution of $\{(R_i, X_i)\}_{i\geqslant 1}$ is homogeneous, identical and independent up to terms of $O(\phi^{1/2})$. More precisely, we assume that the joint probability distribution has a special form which only depends on the one- and two-particle statistics. This closure assumption is motivated by a cluster expansion of the joint probability distribution. The task is to express $\langle B_1|(R_1, X_1), (R_2, X_2)\rangle$ in terms of these one- and two-particle statistics up to an error of $o(\phi^{1/2})$.

We employ a method which allows us to separate screening and correlation effects. The idea is to relate the system with all particles $\{(R_i, X_i)\}_{i\geqslant 1}$ to the system $\{(R_i, X_i)\}_{i\geqslant k+1}$ where k particles have been removed. For instance, we express the Green's function for $\mathbb{R}^3 - \cup_{i\geqslant 1} P_i$ in terms of the Green's function for $\mathbb{R}^3 - \cup_{i\geqslant k} P_i$ up to dipolar terms of $O(\phi^{2/3})$. This amounts to one step in the method of reflection (also called Schwarz alternating method). This deterministic argument captures the screening effects. If $\{(R_i, X_i)\}_{i\geqslant 1}$ are independent, $\{(R_i, X_i)\}_{i\geqslant 1}$ and $\{(R_i, X_i)\}_{i\geqslant k}$ are statistically equivalent in an infinite system. Hence expectations conditioned on the removed particles $\{(R_i, X_i)\}_{i\leqslant k}$ can be replaced by unconditioned expectations. This allows to derive closed equations for conditional expectations.

Our derivation is not rigorous in a strict mathematical sense since we allow ourselves some simplifying assumptions which avoid some technicalities in the computations. The detailed assumptions we make are listed in Section 3.2.

3.1. Statistics

3.1.1. Statistical Setup

A key idea of statistical mechanics is to characterize the deterministic evolution of $\{(R_i, X_i)\}_{i\geqslant 1}$ in statistical terms. Mathematically, this means that one studies the evolution of a probability distribution on $\{(R_i, X_i)\}_{i\geqslant 1}$ under the deterministic dynamics, as described by the Liouville equation. This allows to capture the generic features of the deterministic evolution.

We now set the stage for an infinite, statistically homogeneous system of particles. To avoid technicalities in our calculations, we replace, at a given time t, the infinite system by a periodic system with n particles in a periodic box Ω_n and think of the limit $n\uparrow\infty$. This means that the probability space and the considered random variables depend on n. However, the quantities under consideration (like number densities and expected values of growth rates) will become independent of n in the limit $n\uparrow\infty$. Consequently, for a clearer presentation, we will often omit this dependence in the notation.

The probability distribution of $\{(R_i, X_i)\}_{i\geqslant 1}$ is described by a density function

$$\begin{aligned} &p_n(R_1, X_1, \ldots, R_n, X_n)\, dR_1\, d^3X_1 \cdots dR_n\, d^3X_n \\ &\quad = p_n(1, \ldots, n)\, d(1)\cdots d(n). \end{aligned}$$

It is natural to assume that the distribution is invariant under particle exchange, that is,

$$p_n(\sigma(1), \ldots, \sigma(n)) = p_n(1, \ldots, n) \tag{3.1}$$

for all permutations σ, and invariant under translation, that is,

$$p_n(R_1, X_1 - x, \ldots, R_n, X_n - x) = p_n(R_1, X_1, \ldots, R_n, X_n) \tag{3.2}$$

for all $x \in \mathbb{R}^3$. The probability distribution of $\{(R_i, X_i)\}_{1 \leqslant i \leqslant k}$ can be recovered as

$$p_k(1, \ldots, k) = \int p_n(1, \ldots, n)\, d(k+1) \ldots d(n).$$

Conditional expectations of a random variable $v = v(1, \ldots, n)$ are given by

$$\langle v(1, \ldots, n) \,|\, 1, \ldots, k \rangle = \int v(1, \ldots, n) \frac{p_n(1, \ldots, n)}{p_k(1, \ldots, k)}\, d(k+1) \cdots d(n). \tag{3.3}$$

3.1.2. Number Densities

It is better to characterize the statistics in terms of number densities, since those make sense even in the infinite system limit $n \to \infty$. The one-particle density is given by

$$\begin{aligned} f_1(R) &\overset{(3.2)}{=} f_1(R, X) \\ &= \left\langle \sum_{i=1}^{n} \delta(R - R_i)\delta(X - X_i) \right\rangle \\ &\overset{(3.1)}{=} n\langle \delta(R - R_1)\delta(X - X_1)\rangle \\ &= np_1(R, X). \end{aligned} \tag{3.4}$$

Here and in the following we use the notation e.g. $\overset{(3.1)}{=}$ to indicate that we use Eq. (3.1) to derive the desired equality.

The two-particle density is given by

$$\begin{aligned} f_2(R, \tilde{R}, X - \tilde{X}) &\overset{(3.2)}{=} f_2(R, X, \tilde{R}, \tilde{X}) \\ &= \left\langle \sum_{i=1}^{n} \sum_{j \neq i}^{n} \delta(R - R_i)\delta(X - X_i)\delta(\tilde{R} - R_j)\delta(\tilde{X} - X_j) \right\rangle \\ &\overset{(3.1)}{=} n(n-1)\langle \delta(R - R_1)\delta(X - X_1)\delta(\tilde{R} - R_2)\delta(\tilde{X} - X_2)\rangle \\ &= n(n-1)p_2(R, X, \tilde{R}, \tilde{X}) \end{aligned} \tag{3.5}$$

and so on.

The volume fraction ϕ, the number density ρ, and the capacity density (which defines the screening length ξ) can be expressed in terms of the one-particle density:

$$\begin{aligned}
\phi &= \int \frac{4\pi}{3} R^3 f_1(R)\, dR\,, \\
\rho &= \int f_1(R)\, dR\,, \\
\frac{1}{\xi^2} &= \int 4\pi\, R\, f_1(R)\, dR\,.
\end{aligned} \tag{3.6}$$

In the following we will also use the notation $\langle R \rangle = \langle R_1 \rangle$.

3.1.3. Liouville Equation

Following Marder, we seek to determine the evolution of the one- and two-particle number densities f_1 and f_2. This means that we need to consider the conditional expected values of the growth rates, i.e. $\langle B_1 \mid 1 \rangle$ and $\langle B_1 \mid 1, 2 \rangle$. Indeed, (1.6) translates into the Liouville equations

$$\begin{aligned}
\frac{\partial f_1}{\partial t}(t, R_1) &= \frac{\partial}{\partial R_1}\Big(\frac{1}{R_1^2}\, \langle B_1 \mid 1 \rangle\, f_1(t, R_1)\Big), \\
\frac{\partial f_2}{\partial t}(t, R_1, X_1, R_2, X_2) &= \frac{\partial}{\partial R_1}\Big(\frac{1}{R_1^2}\, \langle B_1 \mid 1, 2 \rangle\, f_2(t, R_1, X_1, R_2, X_2)\Big) \\
&\quad + \frac{\partial}{\partial R_2}\Big(\frac{1}{R_2^2}\, \langle B_2 \mid 1, 2 \rangle\, f_2(t, R_1, X_1, R_2, X_2)\Big).
\end{aligned} \tag{3.7}$$

3.1.4. Statistical Assumptions

In order to close (3.7), one has to express $\langle B_1 \mid 1 \rangle$ and $\langle B_1 \mid 1, 2 \rangle$ in terms of f_1 and f_2, at least up to an error of order $o(\phi^{1/2})$. This requires certain assumptions on the statistics.

Marqusee–Ross assume that the particles are independent:

$$p_n(1, \ldots, n) = \prod_{i=1}^{n} p_1(i). \tag{3.8}$$

Our main goal however is to allow also for correlations between particles. Pair-, triple- and higher correlations in the particle distribution are defined by

$$\begin{aligned}
q_2(1,2) &= p_2(1,2) - p_1(1) p_1(2), \\
q_3(1,2,3) &= p_3(1,2,3) - \big(p_1(1) p_1(2) p_1(3) \\
&\qquad + q_2(1,2) p_1(3) + q_2(2,3) p_1(1) + q_2(1,3) p_1(2)\big)
\end{aligned}$$

and so on. We remark that the joint probability distribution p_n can be expressed in terms of $p_1, q_2, q_3, \ldots$ This is the cluster expansion (ref. 33, Formula (5.4)). Again, in the infinite system limit $n \uparrow \infty$, it is more natural to work with number density based quantities:

$$\begin{aligned} g_2(1,2) &:= n(n-1)q_2(1,2) \\ &= f_2(1,2) - \tfrac{n-1}{n} f_1(1) f_2(2), \\ g_3(1,2,3) &:= n(n-1)(n-2)q_3(1,2,3) \\ &= f_3(1,2,3) - \tfrac{(n-1)(n-2)}{n^2} f_1(1) f_1(2) f_1(3) \\ &\quad - \tfrac{(n-2)}{n}\big(g_2(1,2) f_1(3) + g_2(2,3) f_1(1) + g_2(1,3) f_1(2)\big) \end{aligned} \tag{3.9}$$

and so on.

We now formulate our assumption on the statistics, which are a combination of size and structure assumptions. Following Marder, we assume that pair correlations are of order $\phi^{1/2}$, i.e.

$$\frac{g_2(i,j)}{f_1(i) f_1(j)} = O(\phi^{1/2}), \tag{3.10}$$

and that they are negligible over distances larger than the screening length, that is

$$\frac{g_2(i,j)}{f_1(i) f_1(j)} = o(\phi^{1/2}) \qquad \text{for} \quad |X_i - X_j| \gg \xi. \tag{3.11}$$

These are the size assumptions, now come the structure assumptions. We postulate that higher correlations vanish, i.e. $q_k \equiv 0$ for $k \geqslant 3$, and we neglect products of q_2 in the cluster expansion. This means that we assume the following representation of p_n in terms of p_1 and q_2:

$$p_n(1,\ldots,n) = \prod_{i=1}^{n} p_1(i) + \sum_{i=1}^{n} \sum_{j>i} q_2(i,j) \prod_{k \neq i,j} p_1(k). \tag{3.12}$$

This structure assumption is our closure assumption.

Remark 3.1. Due to the good ergodicity properties enforced by (3.11) the spatial average of u equals the ensemble average – up to an error $o(\phi^{1/2})$ and in the infinite system limit. We hence will assume

$$⨍_{\Omega_n} u(x)\, d^3x = \Big\langle ⨍_{\Omega_n} u(x)\, d^3x \Big\rangle =: u_\infty. \tag{3.13}$$

Furthermore, we treat the following terms as $O(\phi^{1/2})$:

$$\frac{R_1}{\xi} = O(\phi^{1/2}) \qquad \text{and} \qquad \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} = O(\phi^{1/2}).$$

Indeed, recalling (3.6) we find for the expected value of the first term

$$\frac{\langle R\rangle}{\xi} = \frac{\left(4\pi \int R f_1(R)\, dR\right)^{3/2}}{\rho} \leqslant \left(4\pi \int R^3 f_1(R)\, dR\right)^{1/2} = (4\pi\phi)^{1/2}. \tag{3.14}$$

For the second term we notice that only particles within the screening length contribute. The average contribution of particles within the screening length is again $\frac{\langle R\rangle}{\xi} \leqslant (4\pi\phi)^{1/2}$.

3.2. Main Results

3.2.1. One-particle Statistics: The Marqusee–Ross Theory Revisited

We can now state our first result, which is a derivation of the theory first presented by Marqusee and Ross[17] for the one-particle statistics.

Proposition 3.2. Under the assumption (3.1), (3.2) and (3.8) we find in the infinite volume limit

$$\langle B_1\rangle = 0, \tag{3.15}$$

$$\langle B_1 \mid 1\rangle = \left(1 + \frac{R_1}{\xi}\right)(1 - R_1 u_\infty) + o(\phi^{1/2}). \tag{3.16}$$

Remark 3.3. The mean field u_∞ is determined by (3.15) and (3.16) and given by

$$u_\infty = \frac{1 + \frac{\langle R_1\rangle}{\xi}}{\langle R_1\rangle + \frac{\langle R_1^2\rangle}{\xi}} + \frac{1}{\langle R_1\rangle} o(\phi^{1/2}). \tag{3.17}$$

The system (3.7), (3.16) and (3.17) is precisely the Marqusee–Ross theory. The difference with the LSW theory is in the factor $1 + \frac{R_1}{\xi}$ which speeds up the growth or decay of the particle. As opposed to the LSW theory, the evolution equation (3.7) for the one-particle statistics now contains two mean-field type quantities: u_∞ and ξ.

3.2.2. Two-particle Statistics: The Marder Theory Revisited

Our main result in this article is the derivation of an expression for $\langle B_1 \,|\, 1,2\rangle$ under the structure assumption (3.12) of the particle distribution.

Proposition 3.4. Under the assumptions (3.1), (3.2), (3.12), (3.10) and (3.11) we find in the infinite volume limit

$$\langle B_1\rangle = 0,$$

$$\langle B_1\,|1\rangle = \left(1+\frac{R_1}{\xi}\right)(1-R_1(u_\infty+\delta u_1))+o(\phi^{1/2}), \tag{3.18}$$

$$\langle B_1\,|1,2\rangle = \left(1+\frac{R_1}{\xi}\right)(1-R_1(u_\infty+\delta u_1+\delta u_2)) - \frac{R_1}{d_{12}}e^{-\frac{d_{12}}{\xi}}(1-R_2u_\infty) + o(\phi^{1/2}) \ , \tag{3.19}$$

where for $i=1,2$

$$\delta u_i = \int \frac{e^{-\frac{|y-X_1|}{\xi}}}{|y-X_1|}\left(1-Ru_\infty\right)\frac{g_2(R_i,X_i,R,y)}{f_1(R_i)}\,dR\,d^3y = O(\phi^{1/2}). \tag{3.20}$$

Remark 3.5. As in the Marqusee–Ross theory, the mean field u_∞ can be determined from the above relations.

Two conclusions can be drawn from this result

- The last term in (3.19) quantifies how a large particle 2 will negatively affect the growth rate $-B_1$ of particle 1: Particle 1 will grow below average. Hence the large particle 2 over the course of time affects the particle cloud in its neighborhood, as described by (3.7). The quantity $g_2(R_2,X_2,R,y)$ keeps book of this impact.
- This impact leads to the deviation δu_2 in the mesoscopic mean field from its average value u_∞ as described by (3.20). Eq. (3.18) (with particle 1 replaced by particle 2) shows how this in turn influences the growth rate of particle 2.

Let us now address the relation to Marder's work. Up to an implicit term of order $O(\phi)$, (3.18) and (3.19) is identical with Marder's *result* (ref. 16, (2.31)). Our *derivation* differs from Marder's in the initial assumption.

Marder postulates (cf. ref. 16 (2.25)) the following relation between conditional expected charge distributions

$$\left\langle \sum_{i\geqslant 1} B_i\delta(x-X_i)\,|\,1,2\right\rangle = \left\langle \sum_{i\geqslant 1} B_i\delta(x-X_i)\,|\,1\right\rangle + \left\langle \sum_{i\geqslant 1} B_i\delta(x-X_i)\,|\,2\right\rangle.$$

We found this assumption somewhat unsatisfactory, since it is an assumption on the *solution* $\{B_i\}_{i\geqslant 1}$. We replace this assumption on the solution by the assumption on the cluster expansion (3.12), which is an assumption on the *data* $\{(R_i, X_i)\}_{i\geqslant 1}$. Apart from this, we follow the strategy of Marder's inspiring paper.

Remark 3.6. In estimating the error terms we will be rigorous and explicit. However, we allow ourselves the following simplifications:

- We shall neglect dipolar terms. This seems justified since in finite systems they are known to introduce an error of order $O(\phi^{2/3})$.
- We shall assume that particles do not overlap in the strong sense of

$$2(R_i + R_j) \leqslant \min_{j\neq i} |X_i - X_j|. \tag{3.21}$$

- We shall neglect errors due to finite system size.

We shall indicate a place where we neglect one of these error terms by a $\cong$ sign.

In addition, in the proof of Proposition 3.4 we will make the assumption that the deviation of a certain expected value of the potential u from its average is of order $\phi^{1/2}$ (cf. (7.14)). This assumption turns out to be self-consistent in the following sense: under this – relatively weak – assumption on the order of size of the term, we are able to obtain a – rather strong – result, which provides an explicit representation of the term up to an error which is of higher order (cf. Lemma 7.3).

3.2.3. On the Self-consistency of the Statistical Assumptions

So far our analysis only had a snapshot perspective in the sense that we derive our main result Proposition 3.4 under the assumptions on the statistics (3.12), (3.10) and (3.11), which *a priori* are not preserved under the evolution.

We give however an argument that for times of order $\langle R_1\rangle^3$ indeed g_2 remains of order $\phi^{1/2}$ and g_3, even though not vanishing, will remain to be of order $o(\phi^{1/2})$.

Proposition 3.7. The assumptions on the statistics are self-consistent in the sense that

$$\frac{\partial_t g_2(1,2)}{f_1(1)\,f_2(1)} = O\Big(\frac{\phi^{1/2}}{\langle R_1\rangle^3}\Big), \tag{3.22}$$

$$\frac{\partial_t g_2(1,2)}{f_1(1)\,f_2(1)} = o\Big(\frac{\phi^{1/2}}{\langle R_1\rangle^3}\Big), \quad \text{for } \xi \ll |X_1 - X_2| \ll \Big(\frac{n}{\rho}\Big)^{1/3}, \tag{3.23}$$

$$\frac{\partial_t g_3(1,2,3)}{f_1(1)\,f_1(2)\,f_1(3)} = o\Big(\frac{\phi^{1/2}}{\langle R_1\rangle^3}\Big). \tag{3.24}$$

4. THE GREEN'S FUNCTION

The homogenization of the Laplace operator $-\triangle_{\mathrm{D}}$ with Dirichlet boundary conditions on homogeneously distributed holes is by now classical, see e.g. refs. 5, 8 and the references therein. The homogenized operator is the Helmholtz operator $-\triangle + \mu$ where μ is the capacity density of the holes. Both periodic and random arrangements have been considered.

We will present here another derivation of this fact for our random arrangement, including an error estimate. More precisely, we show that the expected value of the Green's function of $-\triangle_{\mathrm{D}}$ agrees with the Green's function of $-\triangle + \mu$ up to an error of $O(\phi^{1/2})$. We will not use this result on the Green's function for the derivation of particle growth rates. However, we find it useful to present the proof since it introduces our strategy on a more elementary level.

To our knowledge our result on the Green's function in infinite systems has not yet been provided in the literature. Random arrangements have been considered for example in ref. 7 or ref. 25, where the fluctuations of the eigenvalues of $-\triangle_{\mathrm{D}}$ around those of $-\triangle + \mu$ are characterized. However, essential to their analysis is the fact that the eigenvalues of $-\triangle_{\mathrm{D}}$ are up to a shift identical to the ones of $-\triangle_{\mathrm{D}} + \alpha$. In finite systems and for sufficiently large α, $(-\triangle_{\mathrm{D}} + \alpha)^{-1}$ is represented by a converging Neumann series similar to (2.10). Our analysis of $-\triangle_{\mathrm{D}}$ in an infinite homogeneous system has to avoid the Neumann series.

Definition 4.1. We denote by $G_j^{(1,\ldots,k)}(x)$ the periodic Green's function of the Laplace operator for the complement of $\cup_{i\geqslant k+1} P_i$ with singularity in X_j, $j \in \{1, \dots, k\}$: i.e.

$$-\Delta_x G_j^{(1,\dots,k)} = 4\pi\delta(\cdot - X_j) \quad \text{outside of} \bigcup_{i\geqslant k+1} P_i,$$
$$G_j^{(1,\dots,k)} = 0 \qquad \text{in} \bigcup_{i\geqslant k+1} P_i.$$

We denote in the following by $H_j^{(1,\dots,k)}(x) := \frac{1}{|x-X_j|} - G_j^{(1,\dots,k)}(x)$ the regular part of the Green's function.

Lemma 4.2. Under the assumptions (3.10)–(3.12) we obtain in the infinite volume limit and up to dipolar terms

$$\langle G_1^{(1)}(x) \,|\, 1\rangle - \frac{1}{|x-X_1|} e^{-\frac{|x-X_1|}{\xi}} = O(\phi^{1/2}) \min\left\{\frac{1}{\xi}, \frac{1}{|x-X_1|}\right\}$$

for all $x \in \mathbb{R}^3 \backslash P_1$.

Proof. Roughly speaking, our claim is that $\langle G_1^{(1)}(x) \,|\, 1\rangle$ is an approximate solution of

$$-\Delta \langle G_1^{(1)}(x) \,|\, 1\rangle + \frac{1}{\xi^2} \langle G_1^{(1)}(x) \,|\, 1\rangle = 4\pi\delta(x - X_1). \tag{4.1}$$

We first give a rough version of the argument, before we show how to control the error.

To this purpose we introduce the charges of $G_1^{(1)}$ on $\{\partial P_i\}_{i\geqslant 2}$ by

$$B_{1,i}^{(1)} := \frac{1}{4\pi} \int_{\partial P_i} \frac{\partial G_1^{(1)}}{\partial \vec{n}}, \tag{4.2}$$

so that up to dipolar terms

$$-\Delta G_1^{(1)}(x) = 4\pi\delta(x - X_1) + \sum_{i\geqslant 2} B_{1,i}^{(1)}\, 4\pi\delta(x - X_i). \tag{4.3}$$

Now comes the first approximation we will control later: The Green's function for the system $\{(R_i, X_i)\}_{i\geqslant 2}$ can be approximated by the Green's function for the reduced system $\{(R_i, X_i)\}_{i\geqslant 3}$ as follows:

$$G_1^{(1)}(x) \approx G_1^{(1,2)}(x) - G_1^{(1,2)}(X_2)\, R_2\, G_2^{(1,2)}(x), \tag{4.4}$$

which in view of (4.2) leads to

$$B_{1,2}^{(1)} \approx -R_2 G_1^{(1,2)}(X_2).$$

Inserting this into (4.3) (with particle 2 replaced by i) already yields a form similar to (4.1):

$$\begin{aligned}
&-\Delta G_1^{(1)}(x) + \sum_{i\geqslant 2} 4\pi R_i\, \delta(x-X_i)\, G_1^{(1,i)}(x) \\
&\quad = -\Delta G_1^{(1)}(x) \ + \sum_{i\geqslant 2} 4\pi R_i\, \delta(x-X_i)\, G_1^{(1,i)}(X_i) \approx 4\pi\,\delta(x-X_1).
\end{aligned}$$

Now it is the right moment to take conditional expectations:

$$\begin{aligned}
&-\Delta\langle G_1^{(1)}(x)|1\rangle + \Big\langle \sum_{i\geqslant 2} 4\pi R_i\, \delta(x-X_i)\, \langle G_1^{(1,2)}(x)\,|\,1,2\rangle|1\Big\rangle \\
&\quad \stackrel{(3.1)}{=} -\Delta\langle G_1^{(1)}(x)\,|\,1\rangle + \Big\langle \sum_{i\geqslant 2} 4\pi R_i\, \delta(x-X_i)\, \langle G_1^{(1,i)}(x)\,|\,1,i\rangle|1\Big\rangle \\
&\quad = -\Delta\langle G_1^{(1)}(x)\,|\,1\rangle + \Big\langle \sum_{i\geqslant 2} 4\pi R_i\, \delta(x-X_i)\, G_1^{(1,i)}(x)|1\Big\rangle \\
&\quad \approx 4\pi\,\delta(x-X_1).
\end{aligned} \tag{4.5}$$

Here comes the second approximation we will quantify later: Since our system is nearly decorrelated, we expect

$$\langle G_1^{(1,2)}(x)\,|\,1,2\rangle \approx \langle G_1^{(1,2)}(x)\,|\,1\rangle\,. \tag{4.6}$$

This allows us to appeal to the following argument: In the infinite volume limit the removal of one particle is immaterial:

$$\langle G_1^{(1,2)}(x)\,|\,1\rangle \approx \langle G_1^{(1)}(x)\,|\,1\rangle\,. \tag{4.7}$$

Inserting (4.6) and (4.7) into (4.5) yields

$$-\Delta\langle G_1^{(1)}(x)\,|\,1\rangle + \Big\langle \sum_{i\geqslant 2} 4\pi R_i\, \delta(x-X_i)\,|\,1\Big\rangle \langle G_1^{(1)}(x)\,|\,1\rangle \approx 4\pi\,\delta(x-X_1). \tag{4.8}$$

Now comes the last approximation to be addressed later: Since our system is nearly decorrelated, we expect

$$\Big\langle \sum_{i\geqslant 2} 4\pi R_i\, \delta(x-X_i)\,|\,1\Big\rangle \approx \Big\langle \sum_{i\geqslant 2} 4\pi R_i\, \delta(x-X_i)\Big\rangle. \tag{4.9}$$

In the infinite volume limit, we have

$$\begin{aligned}\Big\langle \sum_{i\geqslant 2} 4\pi R_i\, \delta(x-X_i)\Big\rangle &\approx \Big\langle \sum_{i\geqslant 1} 4\pi R_i\, \delta(x-X_i)\Big\rangle \\ &\overset{(3.4)}{=} \int 4\pi R_1 \delta(x-X_1) f_1(R_1)\, dR_1\, d^2X_1 \\ &= \int 4\pi R_1\, f_1(R_1)\, dR_1 \\ &\overset{(3.6)}{=} \frac{1}{\xi^2}\,. \end{aligned} \tag{4.10}$$

Inserting (4.9) and (4.10) into (4.8) yields (4.1).

We now show how to control the approximations (4.4), (4.6) and (4.9). Our strategy will be to show

$$-\Delta\langle G_1^{(1)}(x)\,|\,1\rangle + \frac{1}{\xi^2}\langle G_1^{(1)}(x)\,|\,1\rangle = 4\pi\,\delta(x-X_1) + r(x) \tag{4.11}$$

with

$$|r(x)| = O(\phi^{1/2})\frac{1}{\xi^2|x-X_1|}. \tag{4.12}$$

Since $\lim_{|x-X_1|\to\infty}\langle G_1^{(1)}(x)\,|\,1\rangle = 0$, Eq. (4.11) yields

$$\langle G_1^{(1)}(x)\,|\,1\rangle = \frac{1}{|x-X_1|}e^{-\frac{|x-X_1|}{\xi}} + \int \frac{1}{4\pi|x-y|}e^{-\frac{|x-y|}{\xi}}\, r(y)\, d^3y,$$

so that (4.12) entails as desired

$$\begin{aligned}&\Big|\langle G_1^{(1)}(x)\,|\,1\rangle - \frac{1}{|x-X_1|}e^{-\frac{|x-X_1|}{\xi}}\Big| \\ &\leqslant O(\phi^{1/2})\int \frac{1}{\xi^2}\frac{1}{|y-X_1|}\frac{1}{4\pi|x-y|}e^{-\frac{|x-y|}{\xi}}\, d^3y \\ &= O(\phi^{1/2})\min\Big\{\frac{1}{\xi}, \frac{1}{|x-X_1|}\Big\}.\end{aligned}$$

In view of (4.3), the error term $r(x)$ in (4.11) is given by

$$\begin{aligned} r(x) &= -\Big\langle \sum_{i\geqslant 2} B_{1,i}^{(1)}\, 4\pi\,\delta(x-X_i)\,|\,1\Big\rangle - \frac{1}{\xi^2}\langle G_1^{(1)}(x)\,|\,1\rangle \\ &\overset{(3.1)}{=} -(n-1)\langle B_{1,2}^{(1)}\, 4\pi\,\delta(x-X_2)\,|\,1\rangle - \frac{1}{\xi^2}\langle G_1^{(1)}(x)\,|\,1\rangle \\ &= r_1(x)+r_2(x)+r_3(x), \end{aligned}$$

where

$$\begin{aligned} r_1(x) &:= (n-1)\big\langle\big(-B_{1,2}^{(1)} - R_2 G_1^{(1,2)}(X_2)\big)\, 4\pi\,\delta(x-X_2)\,|\,1\big\rangle \\ &= -(n-1)\big\langle\langle B_{1,2}^{(1)} + R_2 G_1^{(1,2)}(X_2)\,|\,1,2\rangle\, 4\pi\,\delta(x-X_2)\,|\,1\big\rangle, \\ r_2(x) &:= (n-1)\big\langle\big(\langle G_1^{(1,2)}(X_2)\,|\,1,2\rangle - \langle G_1^{(1)}(x)\,|\,1\rangle\big)\, 4\pi R_2\,\delta(x-X_2)\,|\,1\big\rangle \\ &= (n-1)\big\langle\big(\langle G_1^{(1,2)}(x)\,|\,1,2\rangle - \langle G_1^{(1)}(x)\,|\,1\rangle\big)\, 4\pi R_2\,\delta(x-X_2)\,|\,1\big\rangle, \\ r_3(x) &:= \langle G_1^{(1)}(x)\,|\,1\rangle\Big((n-1)\langle 4\pi R_2\,\delta(x-X_2)\,|\,1\rangle - \frac{1}{\xi^2}\Big). \end{aligned} \tag{4.13}$$

In estimating r_1, r_2 and r_3 we will be rigorous up to the simplifying assumptions listed in Remark 3.6.

As a preparation for the estimate of r_1, we start by the following estimate on the regular part of the Green's function:

$$\langle H_1^{(1,2)}(X_1)\,|\,1,2\rangle = O\Big(\frac{1}{\xi}\Big). \tag{4.14}$$

Notice that $H_1^{(1,2)} \cong \frac{1}{|X_i-X_1|}$ on ∂P_i for $i\geqslant 3$. Hence, we have by the maximum principle

$$H_1^{(1,2)}(x) \leqslant \sum_{i\geqslant 3;\,|X_i-X_1|\leqslant\xi} \frac{1}{|X_i-X_1|}\,\frac{R_i}{|x-X_i|} + \frac{1}{\xi}. \tag{4.15}$$

Thus

$$\begin{aligned} \langle H_1^{(1,2)}(X_1)\,|\,1,2\rangle &\leqslant \sum_{i\geqslant 3}\Big\langle \chi_{\{|X_1-X_i|\leqslant\xi\}}\frac{R_i}{|X_1-X_i|^2}\,\Big|\,1,2\Big\rangle + \frac{1}{\xi} \\ &\overset{(3.1)}{=} (n-2)\Big\langle \chi_{\{|X_1-X_3|\leqslant\xi\}}\frac{R_3}{|X_1-X_3|^2}\,\Big|\,1,2\Big\rangle + \frac{1}{\xi}. \end{aligned} \tag{4.16}$$

Our structure assumption (3.12) yields in particular

$$p_3(1,2,3) = p_1(1)p_1(2)p_1(3) + q_2(1,2)p_1(3) + q_2(1,3)p_1(2) + q_2(2,3)p_1(1).$$

In view of (3.4) and (3.9), our growth assumption (3.10) can be formulated as

$$\frac{q_2(i,j)}{p_1(i)p_1(j)} = O(\phi^{1/2}) \quad \text{resp.} \quad \frac{p_2(i,j)}{p_1(i)p_1(j)} = 1 + O(\phi^{1/2}). \tag{4.17}$$

We therefore obtain

$$p_3(1,2,3) = (1 + O(\phi^{1/2}))\, p_2(1,2)\, p_1(3).$$

This implies

$$\begin{aligned}
& n\left\langle \chi_{\{|X_1-X_3|\leqslant\xi\}} \frac{R_3}{|X_1-X_3|^2} \,\middle|\, 1,2 \right\rangle \\
& \overset{(3.3)}{=} n \int \chi_{\{|X_1-X_3|\leqslant\xi\}} \frac{R_3}{|X_1-X_3|^2} \, \frac{p_3(1,2,3)}{p_2(1,2)} \, d(3) \\
& = \left(1 + O(\phi^{1/2})\right) n \int \chi_{\{|X_1-X_3|\leqslant\xi\}} \frac{R_3}{|X_1-X_3|^2} \, p_1(3)\, d(3) \\
& \overset{(3.4)}{=} \left(1 + O(\phi^{1/2})\right) \int_{|X_1-X_3|\leqslant\xi} \frac{R_3}{|X_1-X_3|^2} \, f_1(R_3)\, dR_3 d^3X_3 \\
& = \left(1 + O(\phi^{1/2})\right) \xi \int 4\pi R_3 f_1(R_3)\, dR_3 \\
& \overset{(3.6)}{=} \left(1 + O(\phi^{1/2})\right) \frac{1}{\xi}.
\end{aligned} \tag{4.18}$$

Inserting (4.18) into (4.16) entails (4.14).

We now start with r_1. We claim that up to dipolar terms

$$G_1^{(1)}(x) = G_1^{(1,2)}(x) - \frac{R_2 G_1^{(1,2)}(X_2)}{1 - R_2 H_2^{(1,2)}(X_2)} G_2^{(1,2)}(x). \tag{4.19}$$

Indeed, the function

$$v(x) := G_1^{(1)}(x) - G_1^{(1,2)}(x) + \frac{R_2 G_1^{(1,2)}(X_2)}{1 - R_2 H_2^{(1,2)}(X_2)} G_2^{(1,2)}(x)$$

is harmonic outside $\{P_i\}_{i\geqslant 2}$ and in particular near X_1. Furthermore it vanishes on $\{\partial P_i\}_{i\geqslant 3}$. Finally, we have for $x\in\partial P_2$ that

$$\begin{aligned} v(x) &= 0-G_1^{(1,2)}(x)+\frac{R_2 G_1^{(1,2)}(X_2)}{1-R_2 H_2^{(1,2)}(X_2)}G_2^{(1,2)}(x)\\ &\cong 0-G_1^{(1,2)}(X_2)+\frac{R_2 G_1^{(1,2)}(X_2)}{1-R_2 H_2^{(1,2)}(X_2)}\Big(\frac{1}{R_2}-H_2^{(1,2)}(X_2)\Big)\\ &=0\,. \end{aligned}$$

Hence v vanishes up to dipolar terms which establishes (4.19).

Since $G_1^{(1,2)}$ is harmonic near P_2, Eq. (4.19) yields

$$B_{1,2}^{(1)}\cong 0-\frac{R_2 G_1^{(1,2)}(X_2)}{1-R_2 H_2^{(1,2)}(X_2)}\cdot 1$$

and thus

$$B_{1,2}^{(1)}+R_2 G_1^{(1,2)}(X_2)=-\frac{R_2 H_2^{(1,2)}(X_2) R_2 G_1^{(1,2)}(X_2)}{1-R_2 H_2^{(1,2)}(X_2)}. \tag{4.20}$$

We now appeal to the deterministic estimates

$$0\leqslant G_1^{(1,2)}(x)\leqslant\frac{1}{|x-X_1|}, \tag{4.21}$$

$$0\leqslant H_2^{(1,2)}(x)\leqslant\frac{1}{\min_{i\geqslant 3}(|X_2-X_i|-R_i)}, \tag{4.22}$$

which follow from the maximum principle. From (4.22) and (3.21) we infer in particular that

$$2\,R_2\,H_2^{(1,2)}(X_2)\leqslant 1. \tag{4.23}$$

We now use (4.21) (at $x=X_2$) and (4.23) in (4.20). This entails

$$\big|B_{1,2}^{(1)}+R_2 G_1^{(1,2)}(X_2)\big|\leqslant\frac{2R_2^2}{|X_1-X_2|}H_2^{(1,2)}(X_2).$$

Together with (4.14) (where particles 1 and 2 are exchanged) this yields

$$\left|\langle B_{1,2}^{(1)}+R_2 G_1^{(1,2)}(X_2)\,|\,1,2\rangle\right|=O\Big(\frac{1}{\xi}\Big)\frac{R_2^2}{|X_1-X_2|}.$$

We insert this into (4.13)

$$\begin{aligned}|r_1(x)| &\leqslant O\Big(\frac{1}{\xi}\Big)n\Big\langle\frac{R_2^2}{|X_1-X_2|}\,4\pi\,\delta(x-X_2)\,\Big|\,1\Big\rangle\\ &=O\Big(\frac{1}{\xi}\Big)\frac{1}{|X_1-x|}\,n\langle R_2^2\,\delta(x-X_2)\,|\,1\rangle\\ &=O\Big(\frac{1}{\xi}\Big)\frac{1}{|X_1-x|}\frac{n}{p_1(1)}\int R_2^2\,\delta(x-X_2)\big(p_1(1)p_1(2)+q_2(1,2)\big)\,d(2).\end{aligned}$$

Appealing to (4.17) this turns into

$$\begin{aligned}|r_1(x)| &\leqslant O\Big(\frac{1}{\xi}\Big)\frac{1}{|X_1-x|}\,n\int R_2^2\,\delta(x-X_2)\,p_1(2)\,d(2)\\ &= O\Big(\frac{1}{\xi}\Big)\frac{1}{|X_1-x|}\int R_2^2\,\delta(x-X_2)\,f_1(R_2)\,dR_2\,d^3X_2\\ &= O\Big(\frac{1}{\xi}\Big)\frac{1}{|X_1-x|}\int R_2^2\,f_1(R_2)\,dR_2\\ &\leqslant O\Big(\frac{1}{\xi}\Big)\frac{1}{|X_1-x|}\Big(\int R_2\,f_1(R_2)\,dR_2\int R_2^3 f_1(R_2)\,dR_2\Big)^{1/2}\\ &\overset{(3.6)}{=} O\Big(\frac{1}{\xi}\Big)\frac{1}{|X_1-x|}\Big(\frac{1}{\xi^2}\phi\Big)^{1/2}=O(\phi^{1/2})\frac{1}{\xi^2|X_1-x|}.\end{aligned}$$

This establishes (4.12) for the r_1–contribution.

We now address r_2. The first step is to show

$$\left|\langle G_1^{(1,2)}(x)\,|\,1,2\rangle-\langle G_1^{(1,2)}(x)\,|\,1\rangle\right|\leqslant O(\phi^{1/2})\frac{1}{|x-X_1|}. \tag{4.24}$$

To this purpose we use our assumption (3.12) on the structure of the probability distribution to derive the following representation

$$\begin{aligned}&\langle G_1^{(1,2)}(x)\,|\,1,2\rangle-\frac{p_1(1)p_1(2)}{p_2(1,2)}\langle G_1^{(1,2)}(x)\,|\,1\rangle\\ &=\frac{q_2(1,2)}{p_2(1,2)}\int G_1^{(1,2)}(x)\prod_{k\geqslant 3}p_1(k)\,d(k)\\ &\quad+\frac{p_1(1)}{p_2(1,2)}\int G_1^{(1,2)}(x)\sum_{j\geqslant 3}q_2(2,j)\prod_{k\neq 1,2,j}p_1(k)\,d(k)d(j).\end{aligned} \tag{4.25}$$

Indeed, we find

$$
\begin{aligned}
&\langle G_1^{(1,2)}(x)\,|\,1,2\rangle \\
&\overset{(3.3)}{=} \frac{1}{p_2(1,2)}\int G_1^{(1,2)}(x)\,p_n(1,\ldots,n)\prod_{k\geqslant 3} d(k) \\
&\overset{(3.12)}{=} \frac{p_1(1)p_1(2)}{p_2(1,2)}\int G_1^{(1,2)}(x)\prod_{k\geqslant 3} p_1(k)\,d(k) \\
&\quad+\frac{q_2(1,2)}{p_2(1,2)}\int G_1^{(1,2)}(x)\prod_{k\geqslant 3} p_1(k)\,d(k) \\
&\quad+\frac{p_1(1)p_1(2)}{p_2(1,2)}\sum_{3\leqslant i<j}\int G_1^{(1,2)}(x)\,q_2(i,j)\prod_{k\neq 1,2,i,j} p_1(k)\,d(k)d(i)d(j) \\
&\quad+\frac{p_1(2)}{p_2(1,2)}\sum_{j\geqslant 3}\int G_1^{(1,2)}(x)\,q_2(1,j)\prod_{k\neq 1,2,j} p_1(k)\,d(k)d(j) \\
&\quad+\frac{p_1(1)}{p_2(1,2)}\sum_{j\geqslant 3}\int G_1^{(1,2)}(x)\,q_2(2,j)\prod_{k\neq 1,2,j} p_1(k)\,d(k)d(j).
\end{aligned}
$$

Now we use the fact that $G_1^{(1,2)}(x)$ does not depend on particle 2, such that we can multiply the first, third and fourth term with $\int p_1(2)d(2)=1$ to obtain

$$
\begin{aligned}
&\langle G_1^{(1,2)}(x)\,|\,1,2\rangle \\
&\quad=\frac{p_1(1)p_1(2)}{p_2(1,2)}\int G_1^{(1,2)}(x)\prod_{k\geqslant 2} p_1(k)\,d(k) \\
&\qquad+\frac{q_2(1,2)}{p_2(1,2)}\int G_1^{(1,2)}(x)\prod_{k\geqslant 3} p_1(k)\,d(k) \\
&\qquad+\frac{p_1(1)p_1(2)}{p_2(1,2)}\sum_{3\leqslant i<j}\int G_1^{(1,2)}(x)\,q_2(i,j)\prod_{k\neq 1,i,j} p_1(k)\,d(k)d(i)d(j) \\
&\qquad+\frac{p_1(2)}{p_2(1,2)}\sum_{j\geqslant 3}\int G_1^{(1,2)}(x)\,q_2(1,j)\prod_{k\neq 1,j} p_1(k)\,d(k)d(j) \\
&\qquad+\frac{p_1(1)}{p_2(1,2)}\sum_{j\geqslant 3}\int G_1^{(1,2)}(x)\,q_2(2,j)\prod_{k\neq 1,2,j} p_1(k)\,d(k)d(j).
\end{aligned}
$$

On the other hand we find

$$\begin{aligned}
&\langle G_1^{(1,2)}(x) \mid 1\rangle \\
&\quad = \int G_1^{(1,2)}(x) \prod_{k\geqslant 2} p_1(k)\, d(k) \\
&\qquad + \frac{1}{p_1(1)} \int G_1^{(1,2)}(x)\, q_2(1,2) \prod_{k\geqslant 3} p_1(k)\, d(k) d(2) \\
&\qquad + \sum_{3\leqslant i<j} \int G_1^{(1,2)}(x)\, q_2(i,j) \prod_{k\neq 1,i,j} p_1(k)\, d(k) d(i) d(j) \\
&\qquad + \frac{1}{p_1(1)} \sum_{j\geqslant 3} \int G_1^{(1,2)}(x)\, q_2(1,j) \prod_{k\neq 1,j} p_1(k)\, d(k) d(j) \\
&\qquad + \sum_{j\geqslant 3} \int G_1^{(1,2)}(x)\, q_2(2,j) \prod_{k\neq 1,2,j} p_1(k)\, d(k) d(j) d(2).
\end{aligned}$$

Due to $\int q_2(2,j) d(2) = 0$ and the fact that $G_1^{(1,2)}$ does not depend on particle 2 the second and fifth term vanish which in summary yields (4.25).

Since $G_1^{(1,2,j)}(x)$ does not depend on particle j and $\int q_2(2,j)\, d(j) = 0$, Eq. (4.25) can be rewritten as

$$\begin{aligned}
&\langle G_1^{(1,2)}(x) \mid 1,2\rangle - \langle G_1^{(1,2)}(x) \mid 1\rangle \\
&\quad = \frac{q_2(1,2)}{p_2(1,2)} \Big(-\langle G_1^{(1,2)}(x) \mid 1\rangle + \int G_1^{(1,2)}(x) \prod_{k\geqslant 3} p_1(k)\, d(k) \Big) \\
&\qquad + \frac{p_1(1)}{p_2(1,2)} \int \sum_{j\geqslant 3} \big(G_1^{(1,2)}(x) - G_1^{(1,2,j)}(x) \big) q_2(2,j) \prod_{k\neq 1,2,j} p_1(k)\, d(k)\, d(j).
\end{aligned}$$

We now use our assumption on weak correlations in form of (4.17) to conclude from the above

$$\begin{aligned}
&\big| \langle G_1^{(1,2)}(x) \mid 1,2\rangle - \langle G_1^{(1,2)}(x) \mid 1\rangle \big| \\
&\quad \leqslant O(\phi^{1/2}) \Big(\langle G_1^{(1,2)}(x) \mid 1\rangle + \int G_1^{(1,2)}(x) \prod_{k\geqslant 3} p_1(k)\, d(k) \\
&\qquad + \int \sum_{j\geqslant 3} \big| G_1^{(1,2)}(x) - G_1^{(1,2,j)}(x) \big| \prod_{k\geqslant 3} p_1(k)\, d(k) \Big). \qquad (4.26)
\end{aligned}$$

For the last term in (4.26) we appeal to (4.19) (with $G_1^{(1)}$ replaced by $G_1^{(1,2)}$ and particle 2 substituted by particle j) to find

$$G_1^{(1,2)}(x) - G_1^{(1,2,j)}(x) = -G_j^{(1,2,j)}(x) \frac{R_j G_1^{(1,2,j)}(X_j)}{1 - R_j H_j^{(1,2,j)}(X_j)}$$

and to (4.21) and (4.22) in form of

$$\begin{aligned} &0 \leqslant G_1^{(1,2,j)}(X_j) \leqslant \tfrac{1}{|X_1 - X_j|}, \\ &0 \leqslant R_j H_j^{(1,2,j)}(X_j) \leqslant \tfrac{1}{2}, \end{aligned} \tag{4.27}$$

so that

$$\begin{aligned} \sum_{j \geqslant 3} \left| G_1^{(1,2)}(x) - G_1^{(1,2,j)}(x) \right| &\leqslant 2 \sum_{j \geqslant 3} \frac{R_j}{|X_1 - X_j|} G_j^{(1,2,j)}(x) \\ &=: v(x). \end{aligned}$$

We notice that $v(x)$ is harmonic outside $\{P_k\}_{k \geqslant 3}$. For $x \in \partial P_k$ we notice, since $G_j^{(1,2,j)}$ vanishes on ∂P_k for $k \neq j$, that

$$\begin{aligned} v(x) &= 2 \frac{R_k}{|X_1 - X_k|} G_k^{(1,2,k)}(x) \\ &\overset{(4.27)}{\leqslant} 2 \frac{R_k}{|X_1 - X_k|} \frac{1}{|x - X_k|} \\ &= \frac{2}{|X_1 - X_k|} \\ &\cong 2 H_1^{(1,2)}(x). \end{aligned}$$

Thus we have by the maximum principle that

$$v(x) \leqslant 2 H_1^{(1,2)}(x) = 2 \left(\frac{1}{|x - X_1|} - G_1^{(1,2)}(x) \right) \leqslant \frac{2}{|x - X_1|}.$$

Hence we obtain the estimate

$$\sum_{j \geqslant 3} \left| G_1^{(1,2)}(x) - G_1^{(1,2,j)}(x) \right| \leqslant \frac{2}{|x - X_1|}.$$

Using also (4.21), we see that (4.26) turns into (4.24).

Because of (4.7), (4.24) can be written as

$$\left|\langle G_1^{(1,2)}(x)\,|\,1,2\rangle - \langle G_1^{(1)}(x)\,|\,1\rangle\right| \leqslant O(\phi^{1/2})\frac{1}{|x-X_1|},$$

so that

$$|r_2(x)| \leqslant O\big(\phi^{1/2}\big)\frac{1}{|x-X_1|}(n-1)\langle 4\pi R_2\delta(x-X_2)\,|\,1\rangle. \tag{4.28}$$

Once more appealing to $0 \leqslant G_1^{(1)}(x) \leqslant \frac{1}{|x-X_1|}$, we gather

$$|r_3(x)| \leqslant \frac{1}{|x-X_1|}\left|(n-1)\langle 4\pi R_2\delta(x-X_2)\,|\,1\rangle - \frac{1}{\xi^2}\right|. \tag{4.29}$$

Hence (4.28) and (4.29) yield as desired

$$|r_2(x)| + |r_3(x)| \leqslant O\big(\phi^{1/2}\big)\frac{1}{\xi^2|x-X_1|},$$

provided we have

$$\left|(n-1)\langle 4\pi R_2\delta(x-X_2)\,|\,1\rangle - \frac{1}{\xi^2}\right| = O\big(\phi^{1/2}\big)\frac{1}{\xi^2}.$$

Indeed this is easily seen to be true, since

$$\begin{aligned}
&(n-1)\langle 4\pi R_2\delta(x-X_2)\,|\,1\rangle \\
&\quad\cong \frac{n}{p_1(1)}\int 4\pi R_2\delta(x-X_2)\,p_2(1,2)\,d(2) \\
&\quad\overset{(4.17)}{=} \big(1+O(\phi^{1/2})\big)\,n\int 4\pi R_2\delta(x-X_2)\,p_1(2)\,d(2) \\
&\quad\overset{(3.4)}{=} \big(1+O(\phi^{1/2})\big)\int 4\pi R_2\delta(x-X_2)\,f_1(R_2)\,dR_2\,d^3X_2 \\
&\quad= \big(1+O(\phi^{1/2})\big)\int 4\pi R_2 f_1(R_2)\,dR_2 \\
&\quad\overset{(3.6)}{=} \big(1+O(\phi^{1/2})\big)\frac{1}{\xi^2}.
\end{aligned} \tag{4.30}$$

This finishes the proof of the lemma. ∎

5. SCREENING

In this section, we will relate the expectation of the growth rate of particle 1 conditioned on a finite number of particles to the system where these particles have been removed, cf. Lemma 5.2. In analogy to the last section we denote by $u^{(1,\ldots,k)}$ the solution of the elliptic boundary value problem (1.2), (1.3) for the system where particles $1,\ldots,k$ have been removed. The formulas display the screening effect, cf. Remark 5.3. The crucial intermediate step is Lemma 5.1, which will be established by the same strategy as Lemma 4.2.

Lemma 5.1. Under the assumptions (3.10)–(3.12) we find in the infinite volume limit and up to dipolar terms that

$$\begin{aligned}\langle u(x)-u^{(1)}(x)\,|\,1\rangle - \langle B_1\,|\,1\rangle \frac{1}{|x-X_1|} e^{-\frac{|x-X_1|}{\xi}} \\ = O(\phi^{1/2}\ln\phi^{-1}) \min\left\{\frac{1}{\xi}, \frac{1}{|x-X_1|}\right\}\end{aligned} \tag{5.1}$$

for all x outside particle 1. Furthermore, we obtain that

$$\begin{aligned}\langle u(x)-u^{(1,2)}(x)\,|\,1,2\rangle - \langle B_1\,|\,1,2\rangle \frac{1}{|x-X_1|} e^{-\frac{|x-X_1|}{\xi}} \\ -\langle B_2\,|\,1,2\rangle \frac{1}{|x-X_2|} e^{-\frac{|x-X_2|}{\xi}} \\ = O(\phi^{1/2}\ln\phi^{-1}) \min\left\{\frac{1}{\xi}, \max_{i=1,2}\frac{1}{|x-X_i|}\right\}\end{aligned} \tag{5.2}$$

for all x outside particles 1 and 2. Finally, we also have

$$\begin{aligned}\langle u(x)-u^{(1,2,3)}(x)\,|\,1,2,3\rangle - \langle B_1\,|\,1,2,3\rangle \frac{1}{|x-X_1|} e^{-\frac{|x-X_1|}{\xi}} \\ -\langle B_2\,|\,1,2,3\rangle \frac{1}{|x-X_2|} e^{-\frac{|x-X_2|}{\xi}} \\ -\langle B_3\,|\,1,2,3\rangle \frac{1}{|x-X_3|} e^{-\frac{|x-X_3|}{\xi}} \\ = O(\phi^{1/2}\ln\phi^{-1}) \min\left\{\frac{1}{\xi}, \max_{i=1,2,3}\frac{1}{|x-X_i|}\right\}\end{aligned} \tag{5.3}$$

for all x outside particles 1, 2 and 3.

Lemma 5.2. Under the assumptions (3.10)–(3.12) we have

$$\langle B_1 \mid 1\rangle = \Big(1+\frac{R_1}{\xi}\Big)\big(1-R_1\langle u^{(1)}(X_1)\mid 1\rangle\big)+o(\phi^{1/2}), \tag{5.4}$$

$$\begin{pmatrix}\langle B_1 \mid 1,2\rangle\\ \langle B_2 \mid 1,2\rangle\end{pmatrix} = \begin{pmatrix} 1+\frac{R_1}{\xi} & -\frac{R_1}{d_{12}}e^{-\frac{d_{12}}{\xi}}\\ -\frac{R_2}{d_{12}}e^{-\frac{d_{12}}{\xi}} & 1+\frac{R_2}{\xi}\end{pmatrix} \cdot \begin{pmatrix} 1-R_1\,\langle u^{(1,2)}(X_1)\mid 1,2\rangle\\ 1-R_2\,\langle u^{(1,2)}(X_2)\mid 1,2\rangle\end{pmatrix} + o(\phi^{1/2}), \tag{5.5}$$

$$\begin{pmatrix}\langle B_1 \mid 1,2,3\rangle\\ \langle B_2 \mid 1,2,3\rangle\\ \langle B_3 \mid 1,2,3\rangle\end{pmatrix} = \begin{pmatrix} 1+\frac{R_1}{\xi} & -\frac{R_1}{d_{12}}e^{-\frac{d_{12}}{\xi}} & -\frac{R_1}{d_{13}}e^{-\frac{d_{13}}{\xi}}\\ -\frac{R_2}{d_{12}}e^{-\frac{d_{12}}{\xi}} & 1+\frac{R_2}{\xi} & -\frac{R_2}{d_{23}}e^{-\frac{d_{23}}{\xi}}\\ -\frac{R_3}{d_{13}}e^{-\frac{d_{13}}{\xi}} & -\frac{R_3}{d_{23}}e^{-\frac{d_{23}}{\xi}} & 1+\frac{R_3}{\xi}\end{pmatrix} \cdot \begin{pmatrix} 1-R_1\,\langle u^{(1,2,3)}(X_1)\mid 1,2,3\rangle\\ 1-R_2\,\langle u^{(1,2,3)}(X_2)\mid 1,2,3\rangle\\ 1-R_3\,\langle u^{(1,2,3)}(X_3)\mid 1,2,3\rangle\end{pmatrix} + o(\phi^{1/2}). \tag{5.6}$$

Remark 5.3. Like in LSW, that is $B_1 = 1 - R_1 u_\infty$, the formulas in Lemma 5.2 relate the particle growth rate to a mean-field. The new elements are the factors

$$\Big(1+\frac{R_1}{\xi}\Big), \quad \begin{pmatrix} 1+\frac{R_1}{\xi} & -\frac{R_1}{d_{12}}e^{-\frac{d_{12}}{\xi}}\\ -\frac{R_2}{d_{12}}e^{-\frac{d_{12}}{\xi}} & 1+\frac{R_2}{\xi}\end{pmatrix}, \dots,$$

which capture screening. As opposed to the LSW theory, which overestimates the distance between particles, these screening factors reflect the fact that the interaction range is finite and contributes as an amplification factor in the growth rates.

Proof of Lemma 5.1. We only treat (5.1); the identities (5.2) and (5.3) follow analogously. We will first carry out the proof under the assumption that particles are independent, i.e. that (3.8) holds. In the end of the proof we will indicate the main changes which are required under assumptions (3.10)–(3.12).

Our strategy is very similar to Lemma 4.2: We show that $\langle (u - u^{(1)})(x)|1\rangle$ is a solution of

$$\begin{aligned} &-\Delta\langle (u-u^{(1)})(x)\mid 1\rangle + \frac{1}{\xi^2}\,\langle (u-u^{(1)})(x)\mid 1\rangle\\ &= \langle B_1 \mid 1\rangle\, 4\pi\delta(x-X_1) + r \end{aligned} \tag{5.7}$$

and estimate the error term r. We recall the definition of the charges

$$B_i = \frac{1}{4\pi}\int_{\partial P_i} \frac{\partial u}{\partial \vec{n}}, \qquad i \geqslant 1,$$
$$B_i^{(1)} = \frac{1}{4\pi}\int_{\partial P_i} \frac{\partial u^{(1)}}{\partial \vec{n}}, \qquad i \geqslant 2.$$

We have up to dipolar terms

$$-\Delta(u-u^{(1)})(x) = B_1 4\pi\,\delta(x-X_1) + \sum_{i\geqslant 2}(B_i - B_i^{(1)})\,4\pi\,\delta(x-X_i).$$

Taking conditional expectations, this turns into

$$\begin{aligned}&-\Delta\langle (u-u^{(1)})(x)\,|\,1\rangle - (n-1)\big\langle \langle B_2 - B_2^{(1)}\,|\,1,2\rangle\, 4\pi\,\delta(x-X_2)\,|\,1\big\rangle \\ &\quad = \langle B_1\,|\,1\rangle\, 4\pi\,\delta(x-X_1).\end{aligned}$$

Hence the error term in (5.7) is given by

$$r = (n-1)\Big\langle \langle B_2 - B_2^{(1)}|1,2\rangle\, 4\pi\,\delta(x-X_2)\,\Big|\,1\Big\rangle + \frac{1}{\xi^2}\,\langle (u-u^{(1)})(x)\,|\,1\rangle. \qquad (5.8)$$

Since we assume that the particles are independent, we obtain in the infinite volume limit

$$\frac{1}{\xi^2} = n\,\langle 4\pi R_2\delta(x-X_2)\rangle = (n-1)\,\langle 4\pi R_2\delta(x-X_2)\,|\,1\rangle \qquad (5.9)$$

and also

$$\langle (u-u^{(1)})(x)\,|\,1\rangle = \langle (u^{(2)} - u^{(1,2)})(x)\,|\,1,2\rangle \qquad (5.10)$$

so that

$$\frac{1}{\xi^2}\,\langle (u-u^{(1)})(x)\,|\,1\rangle \;= (n-1)\Big\langle \langle R_2(u^{(2)}-u^{(1,2)})(X_2)|1,2\rangle\, 4\pi\,\delta(x-X_2)\,\Big|\,1\Big\rangle.$$

Hence (5.8) turns into

$$r = (n-1)\Big\langle\Big\langle B_2 - B_2^{(1)} + R_2(u^{(2)}-u^{(1,2)})(X_2)\Big|1,2\Big\rangle 4\pi\,\delta(x-X_2)\,\Big|\,1\Big\rangle. \qquad (5.11)$$

Thus we have to relate $B_2 - B_2^{(1)}$ to $-R_2\,(u^{(2)}-u^{(1,2)})(X_2)$.

In the first step we find a suitable representation for $B_2 - B_2^{(1)}$ up to dipolar terms. More precisely, we will show

$$
\begin{aligned}
&B_2 - B_2^{(1)} + R_2(u^{(2)} - u^{(1,2)})(X_2) \\
&\quad = -\frac{(1 - R_1 u^{(1,2)}(X_1))\, R_2 H_2^{(1,2)}(X_2)\, R_2 G_1^{(1,2)}(X_2)}{(1 - R_1 H_1^{(1,2)}(X_1))\,(1 - R_2 H_2^{(1,2)}(X_2))} \\
&\qquad + \frac{(1 - R_2 u^{(2)}(X_2))\, R_2 G_2^{(1,2)}(X_1)\, R_1 G_1^{(1,2)}(X_2)}{(1 - R_1 H_1^{(1,2)}(X_1))\,(1 - R_2 H_2^{(2)}(X_2))\,(1 - R_2 H_2^{(1,2)}(X_2))}.
\end{aligned} \tag{5.12}
$$

The first step towards (5.12) is the formula

$$B_2 = \frac{1 - R_2 u^{(2)}(X_2)}{1 - R_2 H_2^{(2)}(X_2)}. \tag{5.13}$$

In order to show (5.13) we first claim that

$$u(x) - u^{(2)}(x) = B_2 G_2^{(2)}(x). \tag{5.14}$$

Indeed, consider

$$v(x) := u(x) - u^{(2)}(x) - B_2 G_2^{(2)}(x).$$

This function is harmonic outside $\{P_i\}_{i\geqslant 1}$ and vanishes on $\{\partial P_i\}_{i\neq 2}$. Since $u^{(2)}$ is harmonic near X_2, we have

$$\int_{\partial P_2} \frac{\partial v}{\partial \vec{n}} = \int_{\partial P_2} \frac{\partial u}{\partial \vec{n}} - 0 - B_2 \int_{\partial P_2} \frac{\partial G_2^{(2)}}{\partial \vec{n}} = 4\pi B_2 - 4\pi B_2 = 0.$$

Thus, v vanishes up to dipolar terms which establishes (5.14). We evaluate (5.14) at $x \in \partial P_2$ and retain up to dipolar terms

$$\frac{1}{R_2} - u^{(2)}(X_2) = B_2\Big(\frac{1}{R_2} - H_2^{(2)}(X_2)\Big),$$

which turns into (5.13).

The analogue of formula (5.13) also holds for the system $\{(R_i, X_i)\}_{i\geqslant 2}$, that is

$$B_2^{(1)} = \frac{1 - R_2 u^{(1,2)}(X_2)}{1 - R_2 H_2^{(1,2)}(X_2)}. \tag{5.15}$$

From (5.13) and (5.15) we obtain

$$B_2 - B_2^{(1)} = -\frac{R_2(u^{(2)} - u^{(1,2)})(X_2)}{1 - R_2 H_2^{(1,2)}(X_2)} - \frac{(1 - R_2 u^{(2)}(X_2))\, R_2(H_2^{(1,2)} - H_2^{(2)})(X_2)}{(1 - R_2 H_2^{(2)}(X_2))\,(1 - R_2 H_2^{(1,2)}(X_2))}. \tag{5.16}$$

We now appeal to (4.19) with particles 1 and 2 exchanged, which we evaluate at $x = X_2$:

$$(H_2^{(1,2)} - H_2^{(2)})(X_2) = (G_2^{(2)} - G_2^{(1,2)})(X_2) = -G_1^{(1,2)}(X_2)\frac{R_1 G_2^{(1,2)}(X_1)}{1 - R_1 H_1^{(1,2)}(X_1)}.$$

Hence, (5.16) turns into

$$\begin{aligned} B_2 - B_2^{(1)} &= -\frac{R_2(u^{(2)} - u^{(1,2)})(X_2)}{1 - R_2 H_2^{(1,2)}(X_2)} \\ &\quad + \frac{(1 - R_2 u^{(2)}(X_2))\, R_2 G_2^{(1,2)}(X_1)\, R_1 G_1^{(1,2)}(X_2)}{(1 - R_1 H_1^{(1,2)}(X_1))\,(1 - R_2 H_2^{(2)}(X_2))\,(1 - R_2 H_2^{(1,2)}(X_2))} \\ &= -R_2(u^{(2)} - u^{(1,2)})(X_2) \\ &\quad - \frac{R_2(u^{(2)} - u^{(1,2)})(X_2)\, R_2 H_2^{(1,2)}(X_2)}{1 - R_2 H_2^{(1,2)}(X_2)} \\ &\quad + \frac{(1 - R_2 u^{(2)}(X_2))\, R_2 G_2^{(1,2)}(X_1)\, R_1 G_1^{(1,2)}(X_2)}{(1 - R_1 H_1^{(1,2)}(X_1))\,(1 - R_2 H_2^{(2)}(X_2))\,(1 - R_2 H_2^{(1,2)}(X_2))}. \end{aligned} \tag{5.17}$$

In order to obtain (5.12) it remains to reformulate the second term on the right-hand side of (5.17). We appeal to (5.14) with particles 1 and 2 exchanged, and apply it to the system $\{(R_i, X_i)\}_{i \geqslant 2}$:

$$u^{(2)}(x) - u^{(1,2)}(x) = B_1^{(2)} G_1^{(1,2)}(x),$$

and to (5.15) (again with particles 1 and 2 exchanged)

$$B_1^{(2)} = \frac{1 - R_1 u^{(1,2)}(X_1)}{1 - R_1 H_1^{(1,2)}(X_1)}$$

so that

$$R_2(u^{(2)} - u^{(1,2)})(X_2) = \frac{(1 - R_1 u^{(1,2)}(X_1))\, R_2 G_1^{(1,2)}(X_2)}{1 - R_1 H_1^{(1,2)}(X_1)}. \tag{5.18}$$

Hence (5.17) turns into the desired (5.12).

We now will use (5.12) to derive the following deterministic estimate

$$\begin{aligned}
&|B_2 - B_2^{(1)} + R_2(u^{(2)} - u^{(1,2)})(X_2)| \\
&\quad \leqslant 4\,(1 + R_1 u^{(1,2)}(X_1))\, \frac{R_2}{\xi}\, \frac{R_2}{|X_1 - X_2|} \\
&\qquad + 4 \sum_{i \geqslant 3: |X_i - X_2| \leqslant \xi} (1 + R_1 u^{(1,2,i)}(X_1))\, \frac{R_2\, R_i}{|X_i - X_2|^2}\, \frac{R_2}{|X_1 - X_2|} \\
&\qquad + 4 \sum_{i \geqslant 3: |X_i - X_2| \leqslant \xi} R_1\, \frac{R_2}{|X_i - X_2|^2}\, \frac{R_2}{|X_1 - X_2|} \\
&\qquad + 8\,(1 + R_2 u^{(2)}(X_2))\, \frac{R_2}{|X_1 - X_2|}\, \frac{R_1}{|X_1 - X_2|}.
\end{aligned} \tag{5.19}$$

We start by appealing to the estimate (4.21) of the Green's function:

$$0 \leqslant G_i^{(1,2)}(X_j) \leqslant \frac{1}{|X_j - X_i|}, \quad i \neq j \in \{1, 2\}.$$

As in Lemma 4.2, we assume that particles do not touch in the stronger form of (3.21) yielding

$$0 \leqslant R_1 H_1^{(1,2)}(X_1) \leqslant \frac{1}{2}, \quad 0 \leqslant R_2 H_2^{(2)}(X_2) \leqslant \frac{1}{2}, \quad 0 \leqslant R_2 H_2^{(1,2)}(X_2) \leqslant \frac{1}{2}.$$

Hence, (5.12) entails

$$\begin{aligned}
&|B_2 - B_2^{(1)} + R_2(u^{(2)} - u^{(1,2)})(X_2)| \\
&\quad \leqslant 4\,(1 + R_1 u^{(1,2)}(X_1))\, R_2 H_2^{(1,2)}(X_2)\, \frac{R_2}{|X_1 - X_2|} \\
&\qquad + 8\,(1 + R_2 u^{(2)}(X_2))\, \frac{R_2}{|X_1 - X_2|}\, \frac{R_1}{|X_1 - X_2|}.
\end{aligned} \tag{5.20}$$

In order to deduce the optimal stochastic estimate, we have to rewrite the product $u^{(1,2)}(X_1)\,H_2^{(1,2)}(X_2)$. On one hand, we recall (4.15) (with particles 1 and 2 exchanged):

$$H_2^{(1,2)}(X_2) \leqslant \frac{1}{\xi} + \sum_{i\geqslant 3;\,|X_i-X_2|\leqslant\xi} \frac{R_i}{|X_i - X_2|^2}. \tag{5.21}$$

On the other hand, we have by the maximum principle

$$u^{(1,2)}(x) \leqslant u^{(1,2,i)}(x) + \frac{1}{R_i},$$

and thus in particular

$$u^{(1,2)}(X_1) \leqslant u^{(1,2,i)}(X_1) + \frac{1}{R_i}. \tag{5.22}$$

If we insert (5.21) and (5.22) into (5.20), we obtain the desired (5.19).

We will now use (5.19) to deduce the stochastic estimate

$$\begin{aligned}
&\left|\left\langle B_2 - B_2^{(1)} + R_2(u^{(2)} - u^{(1,2)})(X_2)\,\middle|\,1,2\right\rangle\right| \\
&\quad\leqslant 8\left(1 + R_1\,u_\infty + \frac{R_1}{\langle R\rangle}\right)\frac{R_2}{\xi}\,\frac{R_2}{|X_1 - X_2|} \\
&\qquad + 8\,(1 + R_2 u_\infty)\,\frac{R_2}{|X_1 - X_2|}\,\frac{R_1}{|X_1 - X_2|}.
\end{aligned} \tag{5.23}$$

Indeed, we obtain from (5.19)

$$\begin{aligned}
&\left|\left\langle B_2 - B_2^{(1)} + R_2(u^{(2)} - u^{(1,2)})(X_2)\,\middle|\,1,2\right\rangle\right| \\
&\quad\leqslant 4\left(1 + R_1\langle u^{(1,2)}(X_1)|1,2\rangle\right)\frac{R_2}{\xi}\,\frac{R_2}{|X_1 - X_2|} \\
&\qquad + 4\left\langle \sum_{i\geqslant 3:|X_i-X_2|\leqslant\xi} (1 + R_1 u^{(1,2,i)}(X_1))\,\frac{R_i}{|X_i - X_2|^2}\,\middle|\,1,2\right\rangle \frac{R_2^2}{|X_1 - X_2|} \\
&\qquad + 4\left\langle \sum_{i\geqslant 3:|X_i-X_2|\leqslant\xi} \frac{1}{|X_i - X_2|^2}\,\middle|\,1,2\right\rangle \frac{R_1\,R_2^2}{|X_1 - X_2|} \\
&\qquad + 8\left(1 + R_2\langle u^{(2)}(X_2)|1,2\rangle\right)\frac{R_2}{|X_1 - X_2|}\,\frac{R_1}{|X_1 - X_2|}.
\end{aligned} \tag{5.24}$$

We rewrite the two middle terms as

$$\begin{aligned}
&\Big\langle \sum_{i\geqslant 3:|X_i-X_2|\leqslant\xi} (1+R_1u^{(1,2,i)}(X_1))\,\frac{R_i}{|X_i-X_2|^2}\,\Big|\,1,2\Big\rangle \\
&\quad = (n-2)\Big\langle \chi_{\{|X_3-X_2|\leqslant\xi\}}\,(1+R_1\langle u^{(1,2,3)}(X_1)|1,2,3\rangle)\,\frac{R_3}{|X_3-X_2|^2}\,|\,1,2\Big\rangle, \\
&\Big\langle \sum_{i\geqslant 3:|X_i-X_2|\leqslant\xi} \frac{1}{|X_i-X_2|^2}\,\Big|\,1,2\Big\rangle \\
&\quad = (n-2)\Big\langle \chi_{\{|X_3-X_2|\leqslant\xi\}}\,\frac{1}{|X_3-X_2|^2}\,\Big|\,1,2\Big\rangle.
\end{aligned}$$

Because of the assumption of independent particles, we have in the infinite system limit

$$\langle u^{(1,2,3)}(X_1)|1,2,3\rangle = \langle u^{(1,2)}(X_1)|1,2\rangle = \langle u^{(2)}(X_2)|1,2\rangle = u_\infty,$$

so that (5.24) turns into

$$\begin{aligned}
&\Big|\Big\langle B_2 - B_2^{(1)} + R_2(u^{(2)}-u^{(1,2)})(X_2)\Big|1,2\Big\rangle\Big| \\
&\quad \leqslant 4\,(1+R_1u_\infty)\,\frac{R_2}{\xi}\,\frac{R_2}{|X_1-X_2|} \\
&\qquad + 4\,(1+R_1u_\infty)\,(n-2)\Big\langle \chi_{\{|X_3-X_2|\leqslant\xi\}}\frac{R_3}{|X_3-X_2|^2}\,\Big|\,1,2\Big\rangle\,\frac{R_2^2}{|X_1-X_2|} \\
&\qquad + 4\,(n-2)\Big\langle \chi_{\{|X_3-X_2|\leqslant\xi\}}\,\frac{1}{|X_3-X_2|^2}\,\Big|\,1,2\Big\rangle\,\frac{R_1\,R_2^2}{|X_1-X_2|} \\
&\qquad + 8\,(1+R_2u_\infty)\,\frac{R_2}{|X_1-X_2|}\,\frac{R_1}{|X_1-X_2|}.
\end{aligned} \tag{5.25}$$

Because we assume independent particles, we have in the infinite volume limit

$$\begin{aligned}
&(n-2)\Big\langle \chi_{\{|X_3-X_2|\leqslant\xi\}}\frac{R_3}{|X_3-X_2|^2}\,\Big|\,1,2\Big\rangle \\
&\quad = \rho\,\langle R\rangle \int_{|X_3-X_2|\leqslant\xi} \frac{1}{|X_3-X_2|^2}\,d^2X_3 \\
&\quad = \rho\,\langle R\rangle\, 4\pi\xi = \frac{1}{\xi}.
\end{aligned}$$

Likewise, it holds

$$(n-2)\Big\langle \chi_{\{|X_3-X_2|\leqslant\xi\}}\,\frac{1}{|X_3-X_2|^2}\,\Big|\,1,2\Big\rangle = \frac{1}{\langle R\rangle\xi}.$$

Hence (5.25) turns into (5.23).

We now derive the stochastic estimate for the error term r in (5.7). We argue that (5.11) and (5.23) yield

$$|r(x)| \leqslant C\,\phi^{1/2}\Big(1+R_1\,u_\infty+\frac{R_1}{\langle R\rangle}\Big)\Big(\frac{1}{\xi^2\,|x-X_1|}+\frac{1}{\xi\,|x-X_1|^2}\Big). \tag{5.26}$$

Indeed, we deduce from (5.11) and (5.23) that

$$\begin{aligned}|r| \leqslant{}& 8\Big(1+R_1\,u_\infty+\frac{R_1}{\langle R\rangle}\Big)\frac{1}{\xi\,|X_1-X_2|}\,(n-1)\langle R_2^2\,4\pi\,\delta(x-X_2)|1\rangle\\ &+8\,\frac{R_1}{|X_1-X_2|^2}\,(n-1)\langle R_2\,4\pi\,\delta(x-X_2)|1\rangle\\ &+8\,\frac{R_1u_\infty}{|X_1-X_2|^2}\,(n-1)\langle R_2^2\,4\pi\,\delta(x-X_2)|1\rangle.\end{aligned} \tag{5.27}$$

Because we assume independent particles, we have

$$\begin{aligned}&(n-1)\langle R_2^2\,4\pi\,\delta(x-X_2)|1\rangle\\ &\leqslant \Big((n-1)\langle R_2\,4\pi\,\delta(x-X_2)|1\rangle\,(n-1)\langle R_2^3\,4\pi\,\delta(x-X_2)|1\rangle\Big)^{1/2}\\ &\leqslant \Big(n\langle R_2\,4\pi\,\delta(x-X_2)\rangle\,n\langle R_2^3\,4\pi\,\delta(x-X_2)\rangle\Big)^{1/2}\\ &=\Big(\frac{1}{\xi^2}\,3\phi\Big)^{1/2}.\end{aligned}$$

Together with (5.9) we see that (5.27) turns into

$$\begin{aligned}|r(x)| \leqslant C\Big[&\phi^{1/2}\Big(1+R_1\,u_\infty+\frac{R_1}{\langle R\rangle}\Big)\frac{1}{\xi^2\,|x-X_1|}\\ &+\Big(\frac{R_1}{\xi}+\phi^{1/2}R_1u_\infty\Big)\frac{1}{\xi\,|x-X_1|^2}\Big].\end{aligned} \tag{5.28}$$

Using (3.14), we see that (5.28) turns into (5.26).

We now observe that for the fundamental solution $G(y)=\frac{1}{4\pi|y|}e^{-\frac{|y|}{\xi}}$ of $-\Delta+\frac{1}{\xi^2}$ it holds

$$\int_{\mathbb{R}^3}\frac{1}{\xi\,|y-X_1|}G(x-y)d^3y \leqslant C\,\min\left\{1,\frac{\xi}{|x-X_1|}\right\},$$
$$\int_{\mathbb{R}^3}\frac{1}{|y-X_1|^2}G(x-y)d^3y \leqslant C\,\min\left\{\ln\left(1+\frac{\xi}{|x-X_1|}\right),\left(\frac{\xi}{|x-X_1|}\right)^2\right\}$$
$$\leqslant C\,\ln\frac{1}{\phi}\,\min\left\{1,\frac{\xi}{|x-X_1|}\right\},$$

provided $x\notin P_1$. Hence the estimate (5.26) of the error in (5.7) turns into

$$\left|\langle(u-u^{(1)})(x)\,|\,1\rangle-\langle B_1\,|\,1\rangle\frac{1}{|x-X_1|}e^{-\frac{|x-X_1|}{\xi}}\right|$$
$$\leqslant C\left(\phi^{1/2}\,\ln\frac{1}{\phi}\right)\left(1+R_1\,u_\infty+\frac{R_1}{\langle R\rangle}\right)\min\left\{\frac{1}{\xi},\frac{1}{|x-X_1|}\right\}. \quad (5.29)$$

We conclude by a self-consistency argument: As a consequence of Proposition 3.2, we will have $u_\infty=\frac{1}{\langle R\rangle}(1+O(\phi^{1/2}))$, see Remark 3.3. *A fortiori*, this yields the much weaker statement $u_\infty\leqslant C\frac{1}{\langle R\rangle}$. Therefore it is fair to use the latter in (5.29):

$$\left|\langle(u-u^{(1)})(x)\,|\,1\rangle-\langle B_1\,|\,1\rangle\frac{1}{|x-X_1|}e^{-\frac{|x-X_1|}{\xi}}\right|$$
$$\leqslant C\left(\phi^{1/2}\,\ln\phi^{-1}\right)\left(1+\frac{R_1}{\langle R\rangle}\right)\min\left\{\frac{1}{\xi},\frac{1}{|x-X_1|}\right\}$$

To conclude the proof we need to investigate the changes required if we instead of (3.8) assume (3.10)–(3.12). We will control the error in (5.10)

$$\langle(u-u^{(1)})(x)\,|\,1\rangle\cong\langle(u^{(2)}-u^{(1,2)})(x)\,|\,1,2\rangle.$$

We will argue that this error behaves like the one in (5.28). More precisely we will show that

$$\frac{1}{\xi^2}\left|\langle(u-u^{(1)})(x)\,|\,1\rangle-\langle(u^{(2)}-u^{(1,2)})(x)\,|\,1,2\rangle\right|$$
$$\leqslant C\phi^{1/2}\big(1+R_1u_\infty+R_1u_\infty\langle R\rangle u_\infty+R_1\langle u^{(1)}(X_1)\,|\,1\rangle\big)\frac{1}{\xi^2|x-X_1|}. \quad (5.30)$$

This introduces the additional error term $\langle u^{(1)}(X_1)\,|\,1\rangle$. As for u_∞, we will appeal to a self-consistency argument. In the course of the proof of Proposition 3.4 we will show

$$\langle u^{(1)}(X_1)\,|\,1\rangle = u_\infty + \langle v^{(1)}(X_1)\,|\,1\rangle \overset{(7.14)}{=} u_\infty + O(\phi^{1/2})\Big(\frac{1}{\langle R\rangle} + u_\infty\Big).$$

Combining this with $u_\infty = \frac{1}{\langle R\rangle}(1+O(\phi^{1/2}))$, we obtain $\langle u^{(1)}(X_1)\,|\,1\rangle = \frac{1}{\langle R\rangle}(1+O(\phi^{1/2}))$. Thus it seems fair to use the latter in (5.30) to find

$$\frac{1}{\xi^2}\big|\langle (u-u^{(1)})(x)\,|\,1\rangle - \langle (u^{(2)}-u^{(1,2)})(x)\,|\,1,2\rangle\big| \leqslant C\phi^{1/2}\Big(1+\frac{R_1}{\langle R\rangle}\Big)\frac{1}{\xi^2|x-X_1|}.$$

Hence the error from correlations does not exceed the one in (5.28).

We now prove (5.30). In the infinite volume limit, (5.30) can be replaced by

$$\big|\langle (u^{(2)}-u^{(1,2)})(x)\,|\,1\rangle - \langle (u^{(2)}-u^{(1,2)})(x)\,|\,1,2\rangle\big| \leqslant C\phi^{1/2}\big(1+R_1u_\infty + R_1u_\infty\langle R\rangle u_\infty + R_1\langle u^{(1,2)}(X_1)\,|\,1\rangle\big)\frac{1}{|x-X_1|}. \tag{5.31}$$

Since the random variable $(u^{(2)}-u^{(1,2)})(x)$ is independent of particle 2, (4.25) holds with $G_1^{(1,2)}(x)$ replaced by $i\ (u^{(2)}-u^{(1,2)})(x)$. Hence we have analogously to (4.26)

$$\begin{aligned}&\big|\langle u^{(2)}-u^{(1,2)}(x)\,|\,1,2\rangle - \langle u^{(2)}-u^{(1,2)}(x)\,|\,1\rangle\big|\\ &\quad\leqslant C\phi^{1/2}\Big(\langle |u^{(2)}-u^{(1,2)}|(x)\,|\,1\rangle + \int |u^{(2)}-u^{(1,2)}|(x)\prod_{k\geqslant 3}p_1(k)\,d(k)\\ &\qquad+\int\sum_{j\geqslant 3}\big|(u^{(2)}-u^{(1,2)})(x) - (u^{(2,j)}-u^{(1,2,j)})(x)\big|\prod_{k\geqslant 3}p_1(k)\,d(k)\Big).\end{aligned} \tag{5.32}$$

For the first two terms on the right-hand side of (5.32) we use (5.18) with X_2 replaced by x, i.e.

$$(u^{(2)}-u^{(1,2)})(x) = \frac{(1-R_1u^{(1,2)}(X_1))G_1^{(1,2)}(x)}{1-R_1H_1^{(1,2)}(X_1)}, \tag{5.33}$$

which yields the estimate

$$|u^{(2)}-u^{(1,2)}|(x)\leqslant 2(1+R_1u^{(1,2)}(X_1))\frac{1}{|x-X_1|}. \tag{5.34}$$

We therefore obtain for the first term on the right-hand side of (5.32) as desired

$$\langle |(u^{(2)}-u^{(1,2)}|(x)\,|\,1\rangle \leqslant 2\big(1+R_1\langle u^{(1,2)}(X_1)\,|\,1\rangle\big)\frac{1}{|x-X_1|}.$$

For the second term on the right hand side of (5.32) we observe that (5.34) entails

$$\begin{aligned}&\int |u^{(2)}-u^{(1,2)}|(x)\prod_{k\geqslant 3}p_1(k)\,d(k)\\&\leqslant 2\Big(1+R_1\int u^{(1,2)}(X_1)\prod_{k\geqslant 3}p_1(k)\,dk\Big).\end{aligned} \tag{5.35}$$

We now argue that for any y

$$\int u^{(1,2)}(y)\prod_{k\geqslant 3}p_1(k)\,d(k)=u_\infty. \tag{5.36}$$

Indeed we have by translation invariance

$$\begin{aligned}&\int u^{(1,2)}(y)\prod_{k\geqslant 3}p_1(k)\,d(k)\\&=\int\fint_{\Omega_n} u^{(1,2)}(y+h,R_2,X_3+h,\ldots,R_n,X_n+h)\,d^3h\\&\quad\cdot\prod_{k\geqslant 3}p_1(R_k,X_k)\,dR_k\,d^3X_k\\&\overset{(3.2)}{=}\fint_{\Omega_n}\int u^{(1,2)}(y+h,R_2,X_3+h,\ldots,R_n,X_n+h)\\&\quad\cdot\prod_{k\geqslant 3}p_1(R_k,X_k+h)\,dR_k\,d^3X_k\,d^3h\\&=\fint_{\Omega_n}\int u^{(1,2)}(y+h,R_2,X_3,\ldots,R_n,X_n)\\&\quad\cdot\prod_{k\geqslant 3}p_1(R_k,X_k)\,dR_k\,d^3X_k\,d^3h\end{aligned}$$

$$
\begin{aligned}
&= \int \fint_{\Omega_n} u^{(1,2)}(y+h)\,d^3h \prod_{k\geqslant 3} p_1(k)\,d(k) \\
&\overset{(3.13)}{=} \int u_\infty \prod_{k\geqslant 3} p_1(k)\,d(k) \\
&= u_\infty .
\end{aligned}
$$

Thus, also (5.35) can be estimated by the r.h.s. of (5.31).

For the last term in (5.32) we will derive the representation

$$
\begin{aligned}
&\big[(u^{(2)} - u^{(1,2)}) - (u^{(2,j)} - u^{(1,2,j)})\big](x) \\
&\quad = \frac{(1 - R_j u^{(1,2,j)}(X_j)) R_1 G_j^{(1,2,j)}(X_1) G_1^{(1,2,j)}(x)}{(1 - R_j H_j^{(1,2,j)}(X_j))(1 - R_1 H_1^{(1,2,j)}(X_1))} \\
&\qquad - \frac{(1 - R_1 u^{(1,2)}(X_1)) R_j G_1^{(1,2,j)}(X_j) G_j^{(1,2,j)}(x)}{(1 - R_j H_j^{(1,2,j)}(X_j))(1 - R_1 H_1^{(1,2,j)}(X_1))} \\
&\qquad + \frac{(1 - R_1 u^{(1,2)}(X_1)) R_j G_1^{(1,2,j)}(X_j) R_1 G_j^{(1,2,j)}(X_1) G_1^{(1,2)}(x)}{(1 - R_j H_j^{(1,2,j)}(X_j))(1 - R_1 H_1^{(1,2)}(X_1))(1 - R_1 H_1^{(1,2,j)}(X_1))} .
\end{aligned}
\tag{5.37}
$$

Indeed, we combine (5.33) with its version for the system without particle j, i.e.

$$
(u^{(2,j)} - u^{(1,2,j)})(x) = \frac{(1 - R_1 u^{(1,2,j)}(X_1)) G_1^{(1,2,j)}(x)}{1 - R_1 H_1^{(1,2,j)}(X_1)}
$$

to obtain

$$
\begin{aligned}
&\big[(u^{(2)} - u^{(1,2)}) - (u^{(2,j)} - u^{(1,2,j)})\big](x) \\
&\quad = \frac{R_1 (u^{(1,2,j)} - u^{(1,2)})(X_1) G_1^{(1,2,j)}(x)}{1 - R_1 H_1^{(1,2,j)}(X_1)} \\
&\qquad + \frac{(1 - R_1 u^{(1,2)}(X_1))(G_1^{(1,2)} - G_1^{(1,2,j)})(x)}{1 - R_1 H_1^{(1,2,j)}(X_1)} \\
&\qquad + \frac{(1 - R_1 u^{(1,2)}(X_1)) R_1 G_1^{(1,2)}(x)(H_1^{(1,2)} - H_1^{(1,2,j)})(X_1)}{(1 - R_1 H_1^{(1,2)}(X_1))(1 - R_1 H_1^{(1,2,j)}(X_1))} .
\end{aligned}
\tag{5.38}
$$

For the first term on the right-hand side of (5.38) we appeal to (5.33) with particle 1 replaced by particle j and with particle 2 removed, evaluated at $x = X_1$:

$$\big(u^{(1,2)} - u^{(1,2,j)}\big)(X_1) = \frac{(1 - R_j u^{(1,2,j)}(X_j))G_j^{(1,2,j)}(X_1)}{1 - R_j H_j^{(1,2,j)}(X_j)}.$$

For the second term we recall (4.19) with particle 2 replaced by particle j and with particle 2 removed, i.e.

$$(G_1^{(1,2)} - G_1^{(1,2,j)})(x) = -\frac{R_j G_1^{(1,2,j)}(X_j) G_j^{(1,2,j)}(x)}{1 - R_j H_j^{(1,2,j)}(X_j)}. \tag{5.39}$$

For the last term on the right hand side of (5.38) we use (5.39) again, evaluated at $x = X_1$:

$$\begin{aligned}(H_1^{(1,2)} - H_1^{(1,2,j)})(X_1) &= -(G_1^{(1,2)} - G_1^{(1,2,j)})(X_1) \\ &= \frac{R_j G_1^{(1,2,j)}(X_j) G_j^{(1,2,j)}(X_1)}{1 - R_j H_j^{(1,2,j)}(X_j)}.\end{aligned}$$

This establishes (5.37).

The representation (5.37) yields the estimate

$$\begin{aligned}&|(u^{(2)} - u^{(1,2)}) - (u^{(2,j)} - u^{(1,2,j)})|(x) \\ &\quad \leqslant 4\big(1 + R_j u^{(1,2,j)}(X_j)\big) R_1 G_j^{(1,2,j)}(X_1) \frac{1}{|x - X_1|} \\ &\qquad + 4(1 + R_1 u^{(1,2)}(X_1)) R_j \frac{1}{|X_1 - X_j|} G_j^{(1,2,j)}(x) \\ &\qquad + 8(1 + R_1 u^{(1,2)}(X_1)) R_j \frac{1}{|X_1 - X_j|} R_1 G_j^{(1,2,j)}(X_1) \frac{1}{|x - X_1|}.\end{aligned} \tag{5.40}$$

For the two last terms we observe that

$$\sum_{j \geqslant 3} \frac{R_j}{|X_1 - X_j|} G_j^{(1,2,j)}(y) \leqslant H_1^{(1,2)}(y). \tag{5.41}$$

Indeed, the function

$$v(y) := \sum_{j\geqslant 3} \frac{R_j}{|X_1 - X_j|} G_j^{(1,2,j)}(y) - H_1^{(1,2)}(y)$$

is harmonic outside $\cup_{j\geqslant 3} P_j$ and on ∂P_k we have

$$\begin{aligned} v(y) &= \frac{R_k}{|X_1 - X_k|} G_k^{(1,2,k)}(y) - H_1^{(1,2)}(y) \\ &\leqslant \frac{R_k}{|X_1 - X_k|} \frac{1}{|x - X_k|} - \frac{1}{|X_1 - y|} = \frac{1}{|X_1 - X_k|} - \frac{1}{|X_1 - y|} \cong 0. \end{aligned}$$

This proves (5.41).

On one hand we have as in (4.23) that $H_1^{(1,2)}(y) \leqslant \frac{1}{2R_1}$. On the other hand, we have $H_1^{(1,2)}(y) = \frac{1}{|y - X_1|} - G_1^{(1,2)}(y) \leqslant \frac{1}{|y - X_1|}$. Thus, (5.41) entails

$$\sum_{j\geqslant 3} \frac{R_j}{|X_1 - X_j|} G_j^{(1,2,j)}(y) \leqslant \min\left\{\frac{1}{2R_1}, \frac{1}{|y - X_1|}\right\}$$

and (5.40) simplifies to

$$\begin{aligned} &\sum_{j\geqslant 3} |(u^{(2)} - u^{(1,2)}) - (u^{(2,j)} - u^{(1,2,j)})|(x) \\ &\quad \leqslant 4 \sum_{j\geqslant 3} \left(1 + R_j u^{(1,2,j)}(X_j)\right) R_1 G_j^{(1,2,j)}(X_1) \frac{1}{|x - X_1|} \\ &\qquad + 8(1 + R_1 u^{(1,2)}(X_1)) \frac{1}{|x - X_1|}, \end{aligned}$$

so that

$$\begin{aligned} &\int \sum_{j\geqslant 3} |(u^{(2)} - u^{(1,2)}) - (u^{(2,j)} - u^{(1,2,j)})|(x) \prod_{k\geqslant 3} p_1(k)\, d(k) \\ &\quad \leqslant 4 \frac{R_1}{|x - X_1|} \sum_{j\geqslant 3} \int \left(1 + \int R_j u^{(1,2,j)}(X_j) p_1(j)\, d(j)\right) \\ &\qquad \cdot G_j^{(1,2,j)}(X_1) \prod_{\substack{k\geqslant 3 \\ k\neq j}} p_1(k)\, d(k) \\ &\qquad + 8\left(1 + R_1 \int u^{(1,2)}(X_1) \prod_{k\geqslant 3} p_1(k)\, d(k)\right) \frac{1}{|x - X_1|}. \end{aligned} \tag{5.42}$$

We notice that

$$\int R_j u^{(1,2,j)}(X_j)p_1(j)\,d(j) = \langle R\rangle \fint_{\Omega_n} u^{(1,2,j)}\,d^3x = \langle R\rangle u_\infty,$$

so that

$$\begin{aligned}
&\sum_{j\geqslant 3}\int \bigl(1+\int R_j u^{(1,2,j)}(X_j)p_1(j)\,d(j)\bigr)G_j^{(1,2,j)}(X_1)\prod_{\substack{k\geqslant 3\\ k\neq j}} p_1(k)\,d(k)\\
&\quad=\sum_{j\geqslant 3}\int (1+\langle R\rangle u_\infty)G_j^{(1,2,j)}(X_1)\prod_{k\geqslant 3} p_1(k)\,d(k)\\
&\quad=(1+\langle R\rangle u_\infty)\int \sum_{j\geqslant 3} G_j^{(1,2,j)}(X_1)\prod_{k\geqslant 3} p_1(k)\,d(k).
\end{aligned}\tag{5.43}$$

We now observe that

$$\sum_{j\geqslant 3} G_j^{(1,2,j)}(y)\leqslant u^{(1,2)}(y).\tag{5.44}$$

Indeed, the function

$$v(y):=\sum_{j\geqslant 3} G_j^{(1,2,j)}(y)-u^{(1,2)}(y)$$

is harmonic outside $\cup_{j\geqslant 3}P_j$ and on ∂P_k it satisfies

$$v(y)=G_k^{(1,2,k)}(y)-u^{(1,2)}(y)\leqslant \frac{1}{|y-X_k|}-\frac{1}{R_k}=0.$$

This establishes (5.44). Inserting (5.44) into (5.43) turns (5.42) into

$$\begin{aligned}
&\int \sum_{j\geqslant 3} |(u^{(2)}-u^{(1,2)})-(u^{(2,j)}-u^{(1,2,j)})|(x)\prod_{k\geqslant 3} p_1(k)\,d(k)\\
&\quad\leqslant 4(1+\langle R\rangle u_\infty)\frac{R_1}{|x-X_1|}\int u^{(1,2)}(X_1)\prod_{k\geqslant 3} p_1(k)\,d(k)\\
&\qquad+8\Bigl(1+R_1\int u^{(1,2)}(X_1)\prod_{k\geqslant 3} p_1(k)\,d(k)\Bigr)\frac{1}{|x-X_1|}\\
&\quad\overset{(5.36)}{=} 4\bigl(1+\langle R\rangle u_\infty\bigr)R_1u_\infty\frac{1}{|x-X_1|}+8(1+R_1u_\infty)\frac{1}{|x-X_1|}.
\end{aligned}$$

Hence also the last term in (5.32) can be estimated as stated in (5.31). ∎

Proof of Lemma 5.2. We derive Lemma 5.2 from Lemma 5.1. We only show how to get (5.4) from (5.1). To this purpose, we evaluate (5.1) at $x \in \partial P_1$, which yields

$$\begin{aligned}&\big(1 - R_1 \langle u^{(1)}(x) \,|\, 1\rangle\big) - \langle B_1 \,|\, 1\rangle \Big(1 - \frac{R_1}{\xi} + O\Big(\Big(\frac{R_1}{\xi}\Big)^2\Big)\Big)\\ &\quad = O(\phi^{1/2} \ln \phi^{-1}) \min\Big\{\frac{R_1}{\xi}, 1\Big\}.\end{aligned}$$

Following Remark 3.1, we treat $\frac{R_1}{\xi}$ as a term of order $O(\phi^{1/2})$ which gives

$$\big(1 - R_1 \langle u^{(1)}(x) \,|\, 1\rangle\big) - \langle B_1 \,|\, 1\rangle \Big(1 + \frac{R_1}{\xi}\Big)^{-1} = O(\phi \ln \phi^{-1}) = o(\phi^{1/2}).$$

Since $u^{(1)}(x)$ is smooth in particle 1, we can replace $x \in \partial P_1$ by X_1 by introducing only an error of dipolar type. This yields (5.4). ∎

6. ONE-PARTICLE STATISTICS: PROOF OF PROPOSITION 3.2

In this section we rederive the Marqusee–Ross theory, i.e. we prove Proposition 3.2. As opposed to their calculation, our approach allows us to avoid the non-converging series (2.10). Our calculation is indeed closer to Marder's in the sense that we directly calculate finite particle statistics without attempting to invert $\mathbf{1} - \mathbf{g}$. It is simpler than Marder's strategy would be, since Lemma 5.2 allows us to make efficient use of the assumption of statistical independence (3.8).

Proof of Proposition 3.2. It is straight forward to derive (3.16) from formula (5.4) in Lemma 5.2. Indeed, since $u^{(1)}$ does not depend on particle 1, we have

$$\begin{aligned}&\langle u^{(1)}(x) \,|\, 1\rangle\\ &\quad = \langle u^{(1)}(x)\rangle \text{ since particles are statistically independent, cf. (3.8)}\\ &\quad = \langle u(x)\rangle \quad \text{since the infinite systems } \{(R_i, X_i)\}_{i \geqslant 1} \text{ and}\\ &\qquad\qquad \{(R_i, X_i)\}_{i \geqslant 2} \text{ are statistically equivalent,}\\ &\quad = \langle u(0)\rangle \quad \text{due to translation invariance cf. (3.2)}\\ &\quad \overset{(3.13)}{=} u_\infty.\end{aligned}$$

Since particles are identically distributed we obtain (3.15) directly from (1.7). ∎

7. TWO-PARTICLE STATISTICS: PROOF OF PROPOSITION 3.4

In this section, we prove Proposition 3.4, which rederives Marder's theory, under assumptions (3.12), (3.10) and (3.11) on the statistics of the particles.

7.1. A Simple Consequence of the Statistical Assumptions

We introduce the abbreviation $v^{(1,\ldots,k)} := u^{(1,\ldots,k)} - u_\infty$. In the infinite volume limit, v and $v^{(1,\ldots,k)}$ have the same spatial average and we obtain from (3.13)

$$\fint_{\Omega_n} v(x)\, d^3x = 0. \tag{7.1}$$

The following lemma relates different expected values of v.

Lemma 7.1. Under the assumptions (3.10)–(3.12) we have

$$\langle v^{(1,2)}(x)\,|\,1,2\rangle = \Big(\langle v^{(1)}(x)\,|\,1\rangle + \langle v^{(2)}(x)\,|\,2\rangle\Big)(1+O(\phi^{1/2})), \tag{7.2}$$

$$\langle v^{(1,2,3)}(x)\,|\,1,2,3\rangle = \Big(\langle v^{(1,2)}(x)\,|\,1,2\rangle + \langle v^{(1,3)}(x)\,|\,1,3\rangle - \langle v^{(1)}(x)\,|\,1\rangle\Big)(1+O(\phi^{1/2})). \tag{7.3}$$

Proof. By definition (3.3) we have

$$\langle v^{(1,2)}(x)\,|\,1,2\rangle = \frac{1}{p_2(1,2)} \int v^{(1,2)}(x) p_n(1,\ldots,n) \prod_{k\geqslant 3} d(k).$$

We will now use the assumption that p_n is of the form (3.12), which we group into

$$p_n(1,\ldots,n) = p_1(1)\,p_1(2) \prod_{k\geqslant 3} p_1(k) \tag{7.4}$$

$$+q_2(1,2) \prod_{k\geqslant 3} p_1(k) \tag{7.5}$$

$$+p_1(1)\,p_1(2) \sum_{3\leqslant i<j} q_2(i,j) \prod_{k\neq 1,2,i,j} p_1(k) \tag{7.6}$$

$$+p_1(2) \sum_{j\geqslant 3} q(1,j) \prod_{k\neq 1,2,j} p_1(k)$$

$$+p_1(1) \sum_{j\geqslant 3} q(2,j) \prod_{k\neq 1,2,j} p_1(k). \tag{7.7}$$

To compute $\langle v^{(1,2)}(x) \mid 1, 2\rangle$ we integrate with respect to particle $3, \dots, n$. Then the contributions from the terms (7.4), (7.5) and (7.6) vanish because of the following reasons:

- $v^{(1,2)}$ is a function of x and $(R_3, X_3, \dots, R_n, X_n)$ only,
- $v^{(1,2)}$ is a translation invariant function of these variables,
- the statistics are translation invariant (cf. (3.2)),
- the spatial average of $v^{(1,2)}$ vanishes, cf. (7.1).

We give this argument in formulas for the term (7.5) and we make the dependence of $v^{(1,2)}$ on $(R_3, X_3, \dots, R_n, X_n)$ explicit:

$$
\begin{aligned}
&\int v^{(1,2)}(x, R_3, X_3, \dots, R_n, X_n) q_2(1,2) \prod_{k \geqslant 3} p_1(k)\, d(k) \\
&= q_2(1,2) \int v^{(1,2)}(x, R_3, X_3, \dots, R_n, X_n) \prod_{k \geqslant 3} p_1(k)\, d(k) \\
&= q_2(1,2) \int \fint_{\Omega_n} v^{(1,2)}(x+h, R_3, X_3+h, \dots, R_n, X_n+h) \\
&\quad \times \prod_{k \geqslant 3} p_1(k)\, d(k) d^3h \\
&= q_2(1,2) \int \fint_{\Omega_n} v^{(1,2)}(x+h, R_3, X_3, \dots, R_n, X_n) d^3h \prod_{k \geqslant 3} p_1(k)\, d(k) \\
&= 0.
\end{aligned}
$$

Thus we find

$$
\begin{aligned}
&\langle v^{(1,2)}(x) \mid 1, 2\rangle \\
&\quad = \frac{p_1(2)}{p_2(1,2)} \int v^{(1,2)}(x) \sum_{j \geqslant 3} q_2(1, j)\, d(j) \prod_{k \neq 1,2,j} p_1(k)\, d(k) \\
&\qquad + \frac{p_1(1)}{p_2(1,2)} \int v^{(1,2)}(x) \sum_{j \geqslant 3} q_2(2, j)\, d(j) \prod_{k \neq 1,2,j} p_1(k)\, d(k). \qquad (7.8)
\end{aligned}
$$

Since $v^{(1,2)}(x)$ does not depend on particles 1 and 2 we can rewrite the first term in (7.8), using $\int p_1(2)\, d(2) = 1$, and the second, using

$\int p_1(1)\, d(1) = 1$, to obtain

$$\begin{aligned}\langle v^{(1,2)}(x) \mid 1, 2\rangle \\ &= \frac{p_1(2)}{p_2(1,2)} \int v^{(1,2)}(x) \sum_{j\geqslant 3} q_2(1, j)\, d(j) \prod_{k\neq 1, j} p_1(k)\, d(k) \\ &+ \frac{p_1(1)}{p_2(1,2)} \int v^{(1,2)}(x) \sum_{j\geqslant 3} q_2(2, j)\, d(j) \prod_{k\neq 2, j} p_1(k)\, d(k). \qquad (7.9)\end{aligned}$$

We now turn to

$$\langle v^{(1,2)}(x) \mid 1\rangle = \frac{1}{p_1(1)} \int v^{(1,2)}(x)\, p_n(1, \dots, n) \prod_{k\geqslant 2} d(k).$$

As before, the contributions coming from (7.4)–(7.6) vanish. Furthermore, also the contribution coming from (7.7) vanishes due to $\int q_2(2, j)\, d(2) = 0$, yielding

$$\begin{aligned}\langle v^{(1,2)}(x) \mid 1\rangle &= \frac{1}{p_1(1)} \int v^{(1,2)}(x) p_n(1, \dots, n) \prod_{k\geqslant 2} d(k) \\ &= \frac{1}{p_1(1)} \int v^{(1,2)}(x) \sum_{j\geqslant 2} q_2(1, j) d(j) \prod_{k\neq 1, j} p_1(k)\, d(k). \qquad (7.10)\end{aligned}$$

By symmetry, we have

$$\langle v^{(1,2)}(x) \mid 2\rangle = \frac{1}{p_1(2)} \int v^{(1,2)}(x) \sum_{j\geqslant 2} q_2(2, j)\, d(j) \prod_{k\neq 2, j} p_1(k)\, d(k). \qquad (7.11)$$

Combining (7.9), (7.10) and (7.11) we obtain

$$\langle v^{(1,2)}(x) \mid 1, 2\rangle = \frac{p_1(1) p_1(2)}{p_2(1,2)} \big(\langle v^{(1,2)}(x) \mid 1\rangle + \langle v^{(1,2)}(x) \mid 2\rangle\big) \qquad (7.12)$$

We recall (4.17) which gives $\frac{p_1(1)p_1(2)}{p_2(1,2)} = 1 + O(\phi^{1/2})$. In the infinite volume limit the distributions conditioned on particle 1 of $\{(R_i, X_i)\}_{i\geqslant 1}$ and $\{(R_i, X_i)\}_{i\geqslant 2}$ are identical. Thus

$$\langle v^{(1,2)}(x) \mid 1\rangle = \langle v^{(1)}(x) \mid 1\rangle \qquad \text{and} \qquad \langle v^{(1,2)}(x) \mid 2\rangle = \langle v^{(2)}(x) \mid 2\rangle$$

such that we obtain with (3.10) and (4.17) the desired identity (7.2).

To show identity (7.3) we proceed analogously to arrive at

$$\begin{aligned}\langle v^{(1,2,3)}(x)\,|\,1,2,3\rangle = {} & \frac{p_2(1,2)p_1(3)}{p_3(1,2,3)}\langle v^{(1,2,3)}(x)\,|\,1,2\rangle \\ & +\frac{p_2(1,3)p_1(2)}{p_3(1,2,3)}\langle v^{(1,2,3)}(x)\,|\,1,3\rangle \\ & +\frac{p_1(1)p_1(2)p_1(3)}{p_3(1,2,3)}\langle v^{(1,2,3)}(x)\,|\,1\rangle. \qquad (7.13)\end{aligned}$$

Together with (3.10) and (3.12), identity (7.13) implies (7.3). ∎

7.2. Proof of Proposition 3.4

From now on we will assume that

$$R_1\langle v^{(1)}(x)\,|\,1\rangle = O(\phi^{1/2}). \qquad (7.14)$$

Lemma 7.3 below will show that assumption (7.14) is self-consistent. Furthermore Lemma 7.1 implies together with (7.14) that

$$R_1\langle v^{(1,\dots,k)}(x)\,|\,1,\dots,k\rangle = O(\phi^{1/2}) \qquad \text{for} \quad k\leqslant 3. \qquad (7.15)$$

Lemma 7.2. Lemmas 5.1, 5.2, 7.1 and (7.14) imply

$$\langle B_1\,|\,1\rangle = 1 - R_1 u_\infty + O(\phi^{1/2}). \qquad (7.16)$$

$$\langle B_1\,|\,1,2\rangle - \langle B_1\,|\,1\rangle = -R_1\langle v(x)\,|\,2\rangle_{x=X_1} + o(\phi^{1/2}). \qquad (7.17)$$

Proof. We only address (7.17); the argument for (7.16) is similar and simpler. Starting point for (7.17) is Lemma 5.2 in the form of

$$\begin{aligned}\langle B_1\,|\,1\rangle &= \Big(1+\frac{R_1}{\xi}\Big)\big(1-R_1\langle u^{(1)}(X_1)\,|\,1\rangle\big)+o(\phi^{1/2}),\\ \langle B_2\,|\,2\rangle &= \Big(1+\frac{R_2}{\xi}\Big)\big(1-R_2\langle u^{(2)}(X_2)\,|\,2\rangle\big)+o(\phi^{1/2}),\\ \langle B_1\,|\,1,2\rangle &= \Big(1+\frac{R_1}{\xi}\Big)\big(1-R_1\langle u^{(1,2)}(X_1)\,|\,1,2\rangle\big)\\ &\quad -\frac{R_1}{d_{12}}e^{-\frac{d_{12}}{\xi}}\big(1-R_2\langle u^{(1,2)}(X_2)\,|\,1,2\rangle\big)+o(\phi^{1/2}).\end{aligned}$$

These three identities combine to

$$\begin{aligned}
&\langle B_1 \,|\, 1,2\rangle - \langle B_1 \,|\, 1\rangle + \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} \langle B_2 \,|\, 2\rangle \\
&= -\Big(1+\frac{R_1}{\xi}\Big) R_1 \big(\langle u^{(1,2)}(X_1) \,|\, 1,2\rangle - \langle u^{(1)}(X_1) \,|\, 1\rangle\big) \\
&\quad + \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} R_2 \big(\langle u^{(1,2)}(X_2) \,|\, 1,2\rangle - \langle u^{(2)}(X_2) \,|\, 2\rangle\big) \\
&\quad + \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} \frac{R_2}{\xi} \big(1 - R_2 \langle u^{(2)}(X_2) \,|\, 2\rangle\big) + o(\phi^{1/2}).
\end{aligned}$$

We now appeal to (7.2) of Lemma 7.1 which yields

$$\begin{aligned}
&\langle B_1 \,|\, 1,2\rangle - \langle B_1 \,|\, 1\rangle + \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} \langle B_2 \,|\, 2\rangle \\
&= -\Big(1+\frac{R_1}{\xi}\Big) R_1 \langle v^{(2)}(x) \,|\, 2\rangle_{x=X_1} \big(1 + O(\phi^{1/2})\big) \\
&\quad + \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} R_2 \langle v^{(1)}(x) \,|\, 1\rangle_{x=X_2} \big(1 + O(\phi^{1/2})\big) \\
&\quad + \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} \frac{R_2}{\xi} \big(1 - R_2 u_\infty - R_2 \langle v^{(2)}(X_2) \,|\, 2\rangle\big) + o(\phi^{1/2}).
\end{aligned}$$

Thanks to (7.15), this turns into

$$\begin{aligned}
&\langle B_1 \,|\, 1,2\rangle - \langle B_1 \,|\, 1\rangle + \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} \langle B_2 \,|\, 2\rangle \\
&= -R_1 \langle v^{(2)}(x) \,|\, 2\rangle_{x=X_1} + \frac{R_1}{\xi} O(\phi^{1/2}) \\
&\quad + \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} O(\phi^{1/2}) + \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} \frac{R_2}{\xi} O(1) + o(\phi^{1/2}) \\
&= -R_1 \langle v^{(2)}(x) \,|\, 2\rangle_{x=X_1} + o(\phi^{1/2}).
\end{aligned} \tag{7.18}$$

We finally evoke (5.1) in Lemma 5.1 with particle 1 replace by particle 2, i.e.

$$\langle v(x) \,|\, 2\rangle - \langle v^{(2)}(x) \,|\, 2\rangle = \frac{1}{|x - X_2|} e^{-\frac{|x-X_2|}{\xi}} \langle B_2 \,|\, 2\rangle + O\big(\phi^{1/2} \ln \phi^{-1}\big) \frac{1}{\xi}, \tag{7.19}$$

which we evaluate at $x = X_1$ to find

$$
\begin{aligned}
&R_1 \langle v(x) \,|\, 2\rangle_{x=X_1} - R_1 \langle v^{(2)}(x) \,|\, 2\rangle_{x=X_1} \\
&= \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} \langle B_2 \,|\, 2\rangle + O\big(\phi^{1/2} \ln \phi^{-1/2}\big) \frac{R_1}{\xi} \\
&= \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} \langle B_2 \,|\, 2\rangle + o(\phi^{1/2}).
\end{aligned} \tag{7.20}
$$

The combination of (7.18) with (7.20) yields (7.17). ∎

We exploit Lemma 7.2 to find the following central result.

Lemma 7.3. We have in the infinite volume limit that

$$
\begin{aligned}
\langle v(x) \,|\, 1\rangle = &\int \frac{e^{-\frac{|x-y|}{\xi}}}{|x-y|} (1 - R u_\infty) \frac{g_2(R_1, X_1, R, y)}{f_1(R_1)} \, dR \, d^3 y \\
&+ \frac{e^{-\frac{|x-X_1|}{\xi}}}{|x - X_1|} \langle B_1 \,|\, 1\rangle + o\Big(\frac{\phi^{1/2}}{\langle R\rangle}\Big).
\end{aligned}
$$

Proof. The strategy is again to show that $\langle v(x) \,|\, 1\rangle$ satisfies an equation of the form

$$
\begin{aligned}
&-\frac{1}{4\pi} \Delta \langle v(x) \,|\, 1\rangle + \frac{1}{4\pi \xi^2} \langle v(x) \,|\, 1\rangle \\
&= \langle B_1 \,|\, 1\rangle \, \delta(x - X_1) + \int (1 - R u_\infty) \frac{g_2(R_1, X_1, R, x)}{f_1(R_1)} \, dR + r
\end{aligned}
$$

with a controlled error term r. By definition of $\{B_j\}_{j \geqslant 1}$ we have up to dipolar terms that

$$
-\frac{1}{4\pi} \Delta v(x) = \sum_j B_j \, \delta(x - X_j).
$$

We take the expected value conditioned on particle 1:

$$
\begin{aligned}
-\frac{1}{4\pi} \langle \Delta v(x) \,|\, 1\rangle &= \langle B_1 \,|\, 1\rangle \, \delta(x - X_1) + \Big\langle \sum_{j \neq 1} B_j \, \delta(x - X_j) \,\Big|\, 1 \Big\rangle \\
&= \langle B_1 \,|\, 1\rangle \, \delta(x - X_1) + (n-1) \Big\langle \langle B_2 \,|\, 1, 2\rangle \, \delta(x - X_2) \,\Big|\, 1 \Big\rangle.
\end{aligned} \tag{7.21}
$$

The last term in (7.21) has to be identified. To this aim we appeal to Eq. (7.17) in Lemma 7.2 (with particles 1 and 2 exchanged):

$$\langle B_2 \,|\, 1,2\rangle = \langle B_2 \,|\, 2\rangle - R_2 \langle v(x) \,|\, 1\rangle_{x=X_2} + o(\phi^{1/2}).$$

This yields

$$\begin{aligned}
&(n-1)\Big\langle \langle B_2 \,|\, 1,2\rangle\, \delta(x-X_2) \,\Big|\, 1\Big\rangle \\
&= (n-1)\Big\langle \langle B_2 \,|\, 2\rangle\, \delta(x-X_2) \,\Big|\, 1\Big\rangle && (7.22)\\
&\quad -\langle v(x) \,|\, 1\rangle\, (n-1)\, \langle R_2\, \delta(x-X_2) \,|\, 1\rangle && (7.23)\\
&\quad + o(\phi^{1/2})\, (n-1)\, \langle \delta(x-X_2) \,|\, 1\rangle. && (7.24)
\end{aligned}$$

We now address these terms one by one. For (7.24) we notice that

$$\begin{aligned}
(n-1)\, \langle \delta(x-X_2) \,|\, 1\rangle &\overset{(3.10)}{=} n\, \langle \delta(x-X_2)\rangle (1+O(\phi^{1/2}))\\
&= \rho\, (1+O(\phi^{1/2})) = O(\rho). \qquad (7.25)
\end{aligned}$$

For (7.23) we have analogously

$$\begin{aligned}
(n-1)\, \langle R_2\, \delta(x-X_2) \,|\, 1\rangle &\overset{(3.10)}{=} n\, \langle R_2\, \delta(x-X_2)\rangle (1+O(\phi^{1/2}))\\
&= \langle R\rangle\, \rho\, (1+O(\phi^{1/2}))\\
&= \frac{1}{4\pi\xi^2} + O\left(\frac{\phi^{1/2}}{\xi^2}\right). \qquad (7.26)
\end{aligned}$$

In order to handle the $O(\frac{\phi^{1/2}}{\xi^2})$-term in (7.26), we need an estimate on $\langle v(x)|1\rangle$ in (7.23). We will argue that

$$\langle v(x)|1\rangle = O\left(\frac{\phi^{1/2}}{\langle R\rangle} + \frac{1}{|x-X_1|}\right). \qquad (7.27)$$

Indeed, it follows from (7.19) with particle 2 replaced by particle 1

$$\begin{aligned}
\langle v(x) \,|\, 1\rangle &= \langle v^{(1)}(x) \,|\, 1\rangle + \frac{1}{|x-X_1|} e^{-\frac{|x-X_1|}{\xi}} \langle B_1 \,|\, 1\rangle + o\left(\frac{\phi^{1/2}}{\xi}\right)\\
&\overset{(7.14,7.16)}{=} O\left(\frac{\phi^{1/2}}{\langle R\rangle}\right) + \frac{1}{|x-X_1|} e^{-\frac{|x-X_1|}{\xi}} (1 - R_1\, u_\infty + O(\phi^{1/2}))\\
&\quad + o\Big(\frac{\phi^{1/2}}{\xi}\Big)\\
&= O\left(\frac{\phi^{1/2}}{\langle R\rangle}\right) + O\left(\frac{1}{|x-X_1|}\right) + o\left(\frac{\phi^{1/2}}{\xi}\right),
\end{aligned}$$

which yields (7.27)

We now address (7.22) and will argue that

$$(n-1)\langle\langle B_2\,|\,2\rangle\,\delta(x-X_2)\,|\,1\rangle \\ =\int(1-R_2\,u_\infty)\,\frac{g_2(R_1,X_1,R_2,x)}{f_1(R_1)}\,dR_2+O(\phi\rho). \tag{7.28}$$

We introduce the notation $B(R_2):=\langle B_2\,|\,2\rangle$. We recall (3.3), (3.4), (3.5) and use (3.12) to find

$$\begin{aligned}(n-1)&\langle B(R_2)\,\delta(x-X_2)\,|\,1\rangle\\ &=\int B(R_2)\,\delta(x-X_2)\,\frac{f_2(R_1,X_1,R_2,X_2)}{f_1(R_1)}\,dR_2\,d^3X_2\\ &=\int B(R_2)\,\delta(x-X_2)\,f_1(R_2)\,dR_2\,d^3X_2\\ &\quad+\int B(R_2)\,\delta(x-X_2)\,\frac{g_2(R_1,X_1,R_2,X_2)}{f_1(R_1)}\,dR_2\,d^3X_2\\ &=\int B(R_2)\,f_1(R_2)\,dR_2\\ &\quad+\int B(R_2)\,\frac{g_2(R_1,X_1,R_2,x)}{f_1(R_1)}\,dR_2.\end{aligned}$$

However, the first term on the right-hand side vanishes because of (1.7) so that we have

$$(n-1)\langle B(R_2)\,\delta(x-X_2)\,|\,1\rangle=\int B(R_2)\,\frac{g_2(R_1,X_1.R_2,x)}{f_1(R_1)}\,dR_2.$$

We now appeal to (7.16) in Lemma 7.2 and to assumption (3.10) to conclude

$$\begin{aligned}(n-1)&\langle B(R_2)\,\delta(x-X_2)\,|\,1\rangle\\ &=\int(1-R_2\,u_\infty)\,\frac{g_2(R_1,X_1,R_2,x)}{f_1(R_1)}\,dR_2\\ &\quad+O(\phi^{1/2})\int\frac{g_2(R_1,X_1,R_2,x)}{f_1(R_1)}\,dR_2\\ &=\int(1-R_2\,u_\infty)\,\frac{g_2(R_1,X_1,R_2,x)}{f_1(R_1)}\,dR_2+O(\phi)\int f_1(R_2)\,dR_2\\ &=\int(1-R_2\,u_\infty)\frac{g_2(R_1,X_1,R_2,x)}{f_1(R_1)}\,dR_2+O(\phi\rho).\end{aligned} \tag{7.29}$$

This establishes (7.28).

We now collect (7.25)–(7.28) to find

$$
\begin{aligned}
&(n-1)\Big\langle \langle B_2 \,|\, 1,2\rangle\, \delta(x-X_2) \Big|\, 1\Big\rangle \\
&\quad = \int (1-R_2\, u_\infty)\, \frac{g_2(R_1, X_1.R_2, x)}{f_1(R_1)}\, dR_2 - \frac{1}{4\pi\xi^2}\, \langle v(x)\,|\,1\rangle \\
&\qquad + O(\phi\rho) + O\left(\frac{\phi}{\xi^2\langle R\rangle}\right) + O\left(\frac{\phi^{1/2}}{\xi^2|x-X_1|}\right) + o(\phi^{1/2}\rho) \\
&\quad = \int (1-R_2\, u_\infty)\, \frac{g_2(R_1, X_1.R_2, x)}{f_1(R_1)}\, dR_2 - \frac{1}{4\pi\xi^2}\, \langle v(x)\,|\,1\rangle \\
&\qquad + o(\phi^{1/2}\rho) + O\left(\frac{\phi^{1/2}}{\xi^2|x-X_1|}\right).
\end{aligned}
$$

Hence (7.21) turns into

$$
\begin{aligned}
&-\frac{1}{4\pi}\Delta\langle v(x)\,|\,1\rangle + \frac{1}{4\pi\xi^2}\langle v(x)\,|\,1\rangle \\
&\quad = \langle B_1\,|\,1\rangle\, \delta(x-X_1) + \int (1-Ru_\infty)\, \frac{g_2(R_1, X_1, R, x)}{f_1(R_1)}\, dR \\
&\qquad + o(\phi^{1/2}\rho) + O\left(\frac{\phi^{1/2}}{\xi^2|x-X_1|}\right).
\end{aligned}
$$

This implies

$$
\begin{aligned}
\langle v(x)\,|\,1\rangle &= \langle B_1\,|\,1\rangle\, \frac{1}{|x-X_1|} e^{-\frac{|x-X_1|}{\xi}} \\
&\quad + \int \frac{1}{|x-y|} e^{-\frac{|x-y|}{\xi}}\, (1-Ru_\infty)\, \frac{g_2(R_1, X_1, R, y)}{f_1(R_1)}\, dR\, d^3y \\
&\quad + o(\phi^{1/2}\rho\xi^2) + O\left(\frac{\phi^{1/2}}{\xi}\right). \quad \blacksquare
\end{aligned}
$$

Proof of Proposition 3.4. . We start with Lemma 7.3:

$$
\begin{aligned}
\langle v(x)\,|\,1\rangle = \int \frac{e^{-\frac{|x-y|}{\xi}}}{|x-y|} (1-Ru_\infty) \frac{g_2(R_1, X_1, R, y)}{f_1(R_1)}\, dR\, d^3y \\
+ \frac{e^{-\frac{|x-X_1|}{\xi}}}{|x-X_1|} \langle B_1\,|\,1\rangle + o\Big(\frac{\phi^{1/2}}{\langle R\rangle}\Big).
\end{aligned}
$$

We appeal to Lemma 5.1 in form of

$$\begin{aligned}\langle v(x)\,|\,1\rangle - \langle v^{(1)}(x)\,|\,1\rangle &= \langle u(x)\,|\,1\rangle - \langle u^{(1)}(x)\,|\,1\rangle \\ &= \frac{e^{-\frac{|x-X_1|}{\xi}}}{|x-X_1|}\langle B_1\,|\,1\rangle + O\big(\phi^{1/2}\ln\phi^{-1}\big)\frac{1}{\xi}\end{aligned}$$

to conclude, using $\frac{R_1}{\langle R\rangle}=O(1)$ and $\frac{R_1}{\xi}=O(\phi^{1/2})$, that

$$R_1\langle v^{(1)}(x)\,|\,1\rangle = R_1\int \frac{e^{-\frac{|x-y|}{\xi}}}{|x-y|}(1-Ru_\infty)\frac{g_2(R_1,X_1,R,y)}{f_1(R_1)}\,dR\,d^3y + o(\phi^{1/2}).$$

Evaluating this identity at $x=X_1$ yields in the notation of Proposition 3.4:

$$R_1\langle v^{(1)}(X_1)\,|\,1\rangle = R_1\delta u_1 + o(\phi^{1/2}).$$

We now evoke Lemma 5.2 in form of

$$\begin{aligned}\langle B_1\,|\,1\rangle &= \Big(1+\frac{R_1}{\xi}\Big)\big(1-R_1\langle u^{(1)}(X_1)\,|\,1\rangle\big)+o(\phi^{1/2}) \\ &= \Big(1+\frac{R_1}{\xi}\Big)\big(1-R_1u_\infty - R_1\langle v^{(1)}(X_1)\,|\,1\rangle\big)+o(\phi^{1/2})\end{aligned}$$

to obtain

$$\langle B_1\,|\,1\rangle = \Big(1+\frac{R_1}{\xi}\Big)\big(1-R_1u_\infty-R_1\delta u_1\big)+o(\phi^{1/2}), \tag{7.30}$$

which is the first statement of Proposition 3.4.

For the second statement we use Lemma 7.3 with particle 1 replaced by particle 2, i.e.

$$\begin{aligned}\langle v(x)\,|\,2\rangle &= \int \frac{e^{-\frac{|x-y|}{\xi}}}{|x-y|}(1-Ru_\infty)\frac{g_2(R_2,X_2,R,y)}{f_1(R_2)}\,dR\,d^3y \\ &\quad + \frac{e^{-\frac{|x-X_2|}{\xi}}}{|x-X_2|}\langle B_2\,|\,2\rangle + o\Big(\frac{\phi^{1/2}}{\langle R\rangle}\Big),\end{aligned} \tag{7.31}$$

which we evaluate at $x=X_1$:

$$R_1\langle v(x)\,|\,2\rangle_{x=X_1} = R_1\delta u_2 + \frac{R_1}{d_{12}}e^{-\frac{d_{12}}{\xi}}\langle B_2\,|\,2\rangle + o(\phi^{1/2}). \tag{7.32}$$

We now evoke (7.17) in Lemma 7.2, i.e.

$$\langle B_1 \,|\, 1,2\rangle = \langle B_1 \,|\, 1\rangle - R_1 \langle v(x) \,|\, 2\rangle_{x=X_1} + o(\phi^{1/2}), \tag{7.33}$$

to conclude from (7.30) and (7.32) that

$$\begin{aligned}\langle B_1 \,|\, 1,2\rangle &= \Big(1+\frac{R_1}{\xi}\Big)\big(1-R_1 u_\infty - R_1\delta u_1\big) - R_1 \delta u_2 \\ &\quad -\frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} \langle B_2 \,|\, 2\rangle + o(\phi^{1/2}).\end{aligned} \tag{7.34}$$

We use (7.16) in Lemma 7.2 with particle 1 replace by particle 2, i.e.

$$\langle B_2 \,|\, 2\rangle = 1 - R_2 u_\infty + O(\phi^{1/2})$$

and appeal to $R_1\delta u_2 = O(\phi^{1/2})$ to argue that (7.34) turns as desired into

$$\begin{aligned}\langle B_1 \,|\, 1,2\rangle &= \Big(1+\frac{R_1}{\xi}\Big)\big(1-R_1 u_\infty - R_1\delta u_1 - R_1\delta u_2\big) \\ &\quad -\frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} (1-R_2 u_\infty) + o(\phi^{1/2}). \quad \blacksquare\end{aligned}$$

8. PROOF OF THE SELF-CONSISTENCY OF THE STATISTICAL ASSUMPTIONS

It remains to show that our assumptions on the structure of the particle distribution are self-consistent in the sense that they are conserved under the evolution at least over typical time scales.

Proof of Proposition 3.7. We obtain from the definition (3.9) of g_2 and (3.7) that

$$\begin{aligned}\partial_t g_2(1,2) &= \partial_{R_1}\big(R_1^{-2}(\langle B_1 \,|\, 1,2\rangle f_2(1,2) - \langle B_1 \,|\, 1\rangle \tfrac{n-1}{n} f_1(1) f_1(2))\big) \\ &\quad + \partial_{R_2}\big(R_2^{-2}(\langle B_2 \,|\, 1,2\rangle f_2(1,2) - \langle B_2 \,|\, 2\rangle \tfrac{n-1}{n} f_1(1) f_1(2))\big) \\ &= \partial_{R_1}\big(R_1^{-2}(((\langle B_1 \,|\, 1,2\rangle - \langle B_1 \,|\, 1\rangle) f_2(1,2) + \langle B_1 \,|\, 1\rangle g_2(1,2))\big) \\ &\quad + \partial_{R_2}\big(R_2^{-2}(((\langle B_2 \,|\, 1,2\rangle - \langle B_2 \,|\, 2\rangle) f_2(1,2) + \langle B_2 \,|\, 2\rangle g_2(1,2))\big).\end{aligned}$$

Hence, to justify (3.22) and (3.23), we need that $\langle B_1 \,|\, 1,2\rangle - \langle B_1 \,|\, 1\rangle = O(\phi^{1/2})$ and that $\langle B_1 \,|\, 1,2\rangle - \langle B_1 \,|\, 1\rangle = o(\phi^{1/2})$ for $\xi \ll |X_1 - X_2| \ll \big(\frac{n}{\rho}\big)^{1/3}$. But this follows from (7.31) and (7.33).

To show (3.24) we need to invoke (5.6). In fact, a straightforward computation yields

$$\begin{aligned}
&\partial_t g_3(1,2,3)\\
&\quad = \partial_{R_1}\Big(R_1^{-2}(\langle B_1\,|1,2,3\rangle - \langle B_1\,|\,1,2\rangle - \langle B_1\,|\,1,3\rangle + \langle B_1\,|\,1\rangle) f_3(1,2,3)\\
&\qquad + \tfrac{n-2}{n}(\langle B_1\,|\,1,2\rangle - \langle B_1\,|\,1\rangle)(g_2(2,3) f_1(1) + g_2(1,3) f_1(2))\\
&\qquad + \tfrac{n-2}{n}(\langle B_1\,|\,1,3\rangle - \langle B_1\,|\,1\rangle(g_2(2,3) f_1(1) + g_2(1,2) f_1(3))\\
&\qquad + (\langle B_1\,|\,1,2\rangle + \langle B_1\,|\,1,3\rangle - \langle B_1\,|\,1\rangle) g_3(1,2,3)\Big)\\
&\qquad + \partial_{R_2}(\cdots)\\
&\qquad + \partial_{R_3}(\cdots),
\end{aligned}$$

where the terms $(\cdots)$ contain the corresponding symmetric terms to the ∂_{R_1}-term. In view of (3.22), (7.31) and (7.33) it remains to check that

$$\langle B_1\,|1,2,3\rangle - \langle B_1\,|\,1,2\rangle - \langle B_1\,|\,1,3\rangle + \langle B_1\,|\,1\rangle = o(\phi^{1/2}). \tag{8.1}$$

We conclude with (5.4)–(5.6) that up to order $o(\phi^{1/2})$

$$\begin{aligned}
\langle B_1\,|\,1,2,3\rangle = {} & \Big(1+\frac{R_1}{\xi}\Big)(1 - R_1\langle u^{(1,2,3)}(X_1)\,|\,1,2,3\rangle)\\
& -\frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}}(1 - R_2\langle u^{(1,2,3)}(X_2)\,|\,1,2,3\rangle)\\
& -\frac{R_1}{d_{13}} e^{-\frac{d_{13}}{\xi}}(1 - R_3\langle u^{(1,2,3)}(X_3)\,|\,1,2,3\rangle)
\end{aligned} \tag{8.2}$$

and

$$\begin{aligned}
\langle B_1\,|\,1,2\rangle = {} & \Big(1+\frac{R_1}{\xi}\Big)(1 - R_1\langle u^{(1,2)}(X_1)\,|\,1,2\rangle)\\
& -\frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}}(1 - R_2\langle u^{(1,2)}(X_2)\,|\,1,2\rangle),
\end{aligned} \tag{8.3}$$

$$\begin{aligned}
\langle B_1\,|\,1,3\rangle = {} & \Big(1+\frac{R_1}{\xi}\Big)(1 - R_1\langle u^{(1,3)}(X_1)\,|\,1,3\rangle)\\
& -\frac{R_1}{d_{13}} e^{-\frac{d_{13}}{\xi}}((1 - R_3\langle u^{(1,3)}(X_3)\,|\,1,3\rangle),
\end{aligned} \tag{8.4}$$

$$\langle B_1\,|\,1\rangle = \Big(1+\frac{R_1}{\xi}\Big)(1 - R_1\langle u^{(1)}(X_1)\,|\,1\rangle). \tag{8.5}$$

Hence,

$$\begin{aligned}
&\langle B_1 \mid 1,2,3\rangle - \langle B_1 \mid 1,2\rangle - \langle B_1 \mid 1,3\rangle + \langle B_1 \mid 1\rangle \\
&\quad = -\Big(1+\frac{R_1}{\xi}\Big)\big(\langle u^{(1,2,3)}(X_1) \mid 1,2,3\rangle - \langle u^{(1,2)}(X_2) \mid 1,2\rangle \\
&\qquad - \langle u^{(1,3)}(X_3) \mid 1,3\rangle) + \langle u^{(1)}(X_1) \mid 1\rangle) \\
&\qquad - \frac{R_1}{d_{12}} e^{-\frac{d_{12}}{\xi}} R_2(\langle u^{(1,2,3)}(X_2) \mid 1,2,3\rangle - \langle u^{(1,2)}(X_2) \mid 1,2\rangle) \\
&\qquad - \frac{R_1}{d_{13}} e^{-\frac{d_{13}}{\xi}} R_3(\langle u^{(1,2,3)}(X_3) \mid 1,2,3\rangle - \langle u^{(1,3)}(X_3) \mid 1,3\rangle). \qquad (8.6)
\end{aligned}$$

Now the first term on the right-hand side is of order $o(\phi^{1/2})$ due to (7.3) and (7.15). Furthermore

$$\begin{aligned}
&\langle u^{(1,2,3)}(X_2) \mid 1,2,3\rangle - \langle u^{(1,2)}(X_2) \mid 1,2\rangle \\
&\quad = \langle v^{(1,2,3)}(X_2) \mid 1,2,3\rangle - \langle v^{(1,2)}(X_2) \mid 1,2\rangle = O(\phi^{1/2})
\end{aligned}$$

due to (7.15) and thus (8.1) follows from (8.2)–(8.6) and Remark 3.1. ∎

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